

# Asset Demand and Real Interest Rates <sup>\*</sup>

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## Abstract

Understanding factors that drive asset demand is central to understanding movement in long term real interest rates. In this paper we begin by documenting that much of the increase in the demand for asset in the US in the 30 years prior to Covid represented an increased desire to hold more assets for given age and income levels. For example, if we focus on the 55-64 age group, they increased their ratio wealth-to-income by 35-55%. We then develop a model of asset demands which combines retirement motives and inter-temporal substitution motives to quantitatively explore different factor that may have contributed to such a change. Our findings suggest that decreasing interest rates likely led to a substantial increased demand for retirement wealth. We also explore some of the across group heterogeneity and suggest that social security may explain why the lowest income group did not follow the general trend. Finally, we discuss some of the potential macroeconomics implications of long run asset demands that are a decreasing function of interest rates.

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<sup>\*</sup>This paper has benefited from comments and discussion on a previous paper entitled " Gazing at  $r^*$ : A Hysterisis Perspective". While the current paper has a substantially different focus than its early incarnation as " Gazing at  $r^*$ " , we nevertheless thank Sushant Acharya, Tom Carter, Martin Eichenbaum, Jordi Galí, Mark Gertler, Narayana Kocherlakota, Oleksiy Kryvtsov, Guillaume Rocheteau, Ludwig Straub, John Williams and especially Steve Cecchetti, Marcus Hagedorn, Jean-Baptiste Michau, Tim Willems, and Christopher Winter for detailed comments that have lead us reorient our analysis away from intriguing macroeconomic implications (such as multiple steady state real interest rates) and towards a more quantitative analysis of the microeconomic behaviour. We also thank seminar and conference participants at the Bank of Canada, 2022 NBER Summer Institute, 2022 Paris School of Economics Macro Days, 2022 Vienna Macroeconomics Workshop, Swiss National Bank 2022 Annual research conference, CEF 2022, and the University of California, at Irvine for comments. All remaining errors are those of the authors. The views expressed in this paper are those of the authors and not necessarily of the Bank of Canada.

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## 1 Introduction

In most advanced economies, prior to the pandemic, real interest rates had been trending down since the mid to late 1980s (see Figure 1).<sup>1</sup> The most common explanation for this trend is that economies experienced an increased demand for assets that pushed down interest rates and increased the price of other assets, such as stocks and real estate. Important forces cited for inducing such an increase in asset demand include population aging and increased income inequality.<sup>2</sup>

While these factors are certainly relevant, we begin this paper by showing that a key element driving the increased demand for assets over the last thirty years comes from households' desire to hold more assets for given age and income levels. Notably, we document that the increase in the wealth-to-income ratio observed over this period is largely a within group phenomenon as opposed to resulting mainly from changes in demographics or income distribution. Furthermore, we show that saving behavior supports an interpretation of the observed higher wealth holdings as reflecting a desire to hold more wealth as opposed to reflecting wealth levels that are temporarily above target levels due to valuation effects.

When looking to explain why households with similar income and demographic characteristics may have increased their desired asset-to-income ratios over this period, many explanations are possible. Retirement needs in a low interest rate environment is one such possibility.<sup>3</sup> This is nicely expressed by Raghuram Rajan<sup>4</sup>, former governor of the Reserve Bank of India: <sup>5</sup>

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<sup>1</sup>The influential empirical study by [Laubach and Williams \(2003\)](#) provides estimates showing that the natural rate of interest  $r^*$  has been declining.

<sup>2</sup>A vast literature examines the sources of the decreasing trend in real interest rates. [Borio et al. \(2017\)](#) provide an excellent survey of the literature on these issues. Several hypotheses about these sources have been proposed: demographics ([Summers \(2014\)](#), [Eggertsson and Mehrotra \(2014\)](#), [Eichengreen \(2015\)](#)), and [Goodhart and Pradham \(2020\)](#); a productivity slowdown ([Gordon \(2017\)](#)); a global saving glut and/or lack of safe assets ([Bernanke \(2005\)](#), [Caballero et al. \(2008\)](#), [Gourinchas et al. \(2020\)](#), and [Acharya and Dogra \(2022\)](#)); a decline in desired investment ([Rachel and Smith \(2017\)](#)); a rise in inequality ([Mian et al. \(2020\)](#), [Auclert and Rognlie \(2020\)](#), [Fagereng et al. \(2019\)](#), and [Rachel and Smith \(2017\)](#)).

<sup>3</sup>While we focus on retirement motives to help explain asset holdings, bequest motives likely play a similar role. See [Beaudry and Meh \(2021\)](#).

<sup>4</sup>See [Rajan \(2013\)](#).

<sup>5</sup>The link between low interest rates and the need for more retirement savings is also often mentioned in the financial industry, as illustrated by the following examples. In the issue from September 2016 –dedicated to living in a low-rate environment– the Economist's briefing on pensions (<https://www.economist.com/briefing/2016/09/24/fade-to-grey>) noted that investors who had to buy their own pensions knew that low level of interest rates and bond yields meant a higher cost of pensions. According to Moneyfacts, a British data firm, in the late 1990s, £100,000 would have bought a 65-year-old British man a lifelong income of £11,170 a year; while more recently it would earn £4,960. In other words, paying out a given level of income now costs more than twice as much as it did. Similarly, in January 2022, MoneyRates website (<https://www.moneyrates.com/investment/how->

*“...savers put more money aside as interest rates fall in order to meet the savings they think they will need when they retire.”*

With this in mind, this paper develops a model of asset accumulation in a continuous time overlapping generations (OLG) environment that allows for inter-temporal substitution and retirement motives to compete. The model builds on [Blanchard \(1985\)](#) and [Yaari \(1965\)](#), and is closest to [Gertler \(1999\)](#).<sup>6</sup> The framework is sufficiently tractable to allow the relationship between desired asset-to-income ratios and interest rates to be derived analytically. In particular, we show that if the inter-temporal elasticity of substitution is less than 1 (which is the more empirically plausible case), then long-run asset demands become C-shaped, with lower interest rates motivating households to increase their asset-to-income ratios.

We then use this framework to quantitatively explore how different forces may have led to the observed increase in wealth-to-income ratios. This includes the role of increased longevity, decreased aggregate growth and decreased interest rates. While all these factors may have played a role, the decreases in interest rates appears— from our models perspective— as the most likely candidate. In particular, if one believes in an elasticity of inter-temporal substitution slightly below .5 (ie a risk aversion parameter slightly above 2), then the observed increase in within group wealth can be readily explained. Moreover, when quantifying the model, it implies retirement incentives start to dominate inter-temporal substitution motive when interest rate fall somewhat below 3%, imply a C-shaped asset demand function with an inflexion point of around 3%. Accordingly, this suggests that the economy may have been operating on a segment of the asset demand function— where asset demand increases as interest rate falls— for much of the last 30 years.

Much of our quantitative analysis focuses on explaining the observed change in wealth holding of the 55-64 age group. While most of the income groups within this age cohort increased their wealth holding, the lower income groups— those below 70,000\$ per year— appear to have decreased their wealth. We argue that this pattern may reflect the different effect of social security on different income groups. For the lower income groups, social

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[lower-interest-rates-ruin-retirement.htm](#)) suggested to its readers that meeting the same retirement goal with reduced rates of return meant that they had to set more of their paychecks aside. Finally, the Office of Retirement and Disability Policy of the Social Security Administration in its Perspectives series on “Retirement Implications of a Low Wage Growth, Low Real Interest Rate Economy” in 2020 (<https://www.ssa.gov/policy/docs/ssb/v80n3/v80n3p31.html>) provided the following calculation for the increase in the amount of assets needed to fund a retirement goal with a lower real interest rate: “suppose the goal is to accumulate 100,000 after 10 years. With a 3 percent interest rate, 8,469 in annual saving is required. If interest rates fall to 1 percent, annual saving must increase to 9,463 to fund the 100,000 goal. That represents almost a 12 percent bump up in annual saving”.

<sup>6</sup>[Galí \(2021\)](#) introduces retirement in a similar fashion in a New Keynesian model with bubbles.

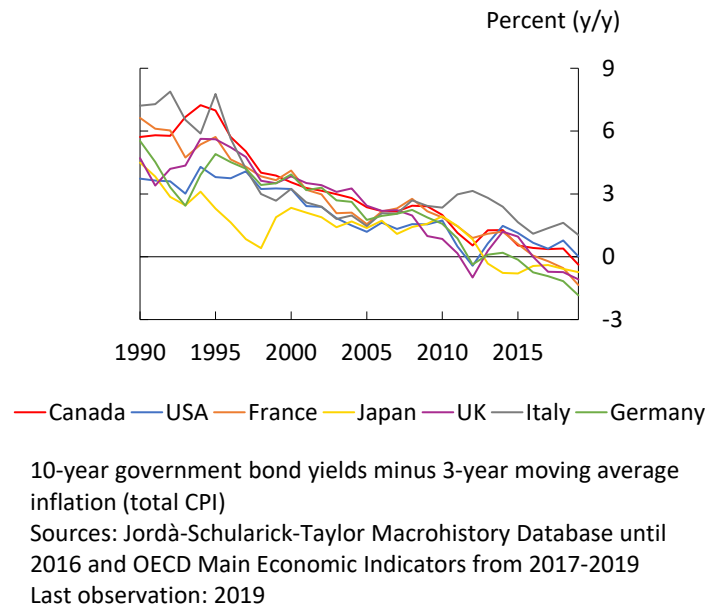


Figure 1  
 Long-term interest rates for G7 countries from 1990 to 2019

security covers much of their retirement needs and therefore a fall in interest rates creates mainly an inter-temporal substitution motive favoring consumption over savings. Hence, we argue that the fall in real interest can explain both why— on a wealth weighted basis— household have accumulated more wealth, while at the same time explaining why lower income individual may have decreased their wealth holdings.

In the last section, we briefly discuss some of the general equilibrium implications of being on the lower portion of a C-shaped assets demand curve. In particular, we discuss how in such an environment ( 1) a reduction in the supply of safe assets can favour an asset boom sufficient to increase overall wealth (2) an small exogenous increase in asset demand can be strong amplified.

The remainder of the paper is organized as follows. Section 2 exploits household level data to examine how asset positions changed over the 30 years prior to the pandemic. We show that for households with similar income and age, asset holding increased substantially. This remains true when removing housing wealth. Towards the end of this section we focus more particularly on the asset holds on the 54-65 age group, and discuss some of the heterogeneity across income groups. In particular, we document how most income groups above 60-70 000\$ increased their wealth-to-income-ratio, while those below 70 000 we see a decrease. Section ?? presents an OLG model—similar in spirit to that of Gertler (1999)—

that integrates both inter-temporal substitution forces and retirement preoccupations. The model is sufficiently tractable to offer an analytic expression for desired wealth holdings. The framework implies that desired wealth holding of pre-retirement individuals become a C-shaped function of real interest rates when the elasticity of inter-temporal elasticity of substitution is below 1. Section 3 uses the model developing in Section ?? to quantitatively explore the potential strength of different forces in influencing asset demand. We begin by focusing on explaining average changes. We find that increases in longevity and decreased expected growth cannot explain the size of the observed (average) within group change. In contrast, we find that the observed decrease in real interest rates is easily capable of explaining the size of the observed change when adopting commonly estimated values for the elasticities of inter-temporal elasticity of substitution. Section ?? discuss some of the potential macroeconomic implications of our findings. and Section 5 concludes.

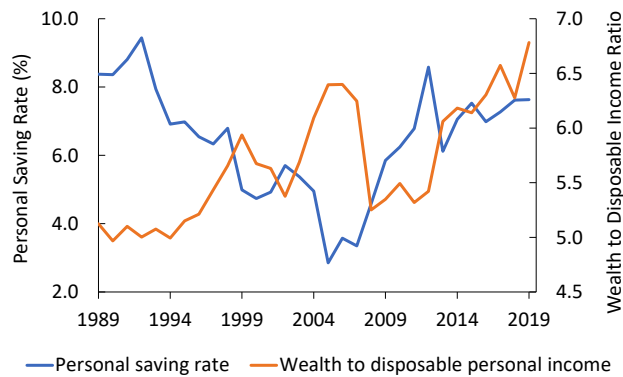
## 2 A Between versus Within Household Decomposition of Aggregate Asset Holdings over 30 years: 1989-2019

Figure 2 illustrates how the aggregate wealth-to-income ratio in the US increased significantly and the aggregate saving rate decreased only very mildly over the 1989-2019 period.<sup>7</sup> The question we want to address is how best to interpret and quantify such observation; should it mainly be interpreted as reflecting between group (composition) effects or do it instead largely reflect within group choices.

**Between group (compositional effect) explanation.** A common explanation for the rise in aggregate wealth-to-income ratio is that it reflects increase in demand for assets induced by changes in demographics and income distribution. As the population aged, and more income was concentrated in higher income groups, the demand for wealth increased. This put downward pressure on interest rates, which through valuation effects among others, raised the effective supply of wealth. The higher savings of the older and richer population was compensated by a decreased incentive to save by the population at large due to lower interest rates, leaving the overall savings rate relatively flat. Such narrative is essentially a “between” group narrative which relies on compositional changes in types of individuals to explain the increased demand for wealth.

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<sup>7</sup>Mian et al. (2021b) provide an extensive analysis of the evolution of household saving behavior using the ratio of saving to national income since 1950, including over the period from 1995 to 2019, when the natural rate of interest fell to an extremely low level.



Sources: U.S. BEA for Personal Saving Rate and Disposable Personal Income; Board of Governors of the Federal Reserve System for Wealth. Retrieved from FRED.  
 Last Observation: 2019

Figure 2  
 Household saving rates and aggregate wealth-to-income ratios in the US from 1989 to 2019

**Within group explanation.** At the other end of the spectrum, it is possible that within group demand for asset played a significant role in the overall change in the demand for wealth. This could reflect many forces including increased longevity, decrease expected growth or the effects of decreases in interest rates on asset demand.

The above discussion underlines the relevance of understanding the relative roles of within versus between group effects in explaining the increased wealth holdings in the US over the last three decades. To do so, we use the Survey of Consumer Finances (SCF) and focus on the difference in asset holdings across household groups between 1989 and 2019. We choose this period for our analysis as it corresponds quite closely to the period of decreasing real interest rates presented in Figure 1. Furthermore, by looking at this thirty-year difference, we hope to minimize higher frequency movements in wealth accumulation dynamics associated with business cycles forces and crises.

The SCF is the most comprehensive source of data on household-level wealth and its components in the United States. It also has a consistent sampling methodology, over-sampling the rich, in all the survey waves between 1989 and 2019, which is useful for our analysis. The survey has between 3 and 5.5 thousand households, depending on the year, and our results use weights throughout. For our baseline definition of wealth, given the importance of retirement considerations in the theory sections, we supplement the SCF data with the estimates on defined benefit (DB) pensions of households from [Sabelhaus](#)

and Volz (2020), which have been widely adopted in the related literature.<sup>8</sup> Thus, our measurement of wealth, including DB pension wealth and excluding social security wealth, also lines up well with that reported in Financial Accounts of the Federal Reserve (FA).<sup>9</sup> In this section, we primarily report findings using the SCF (plus DB pensions) data, which allows us to establish our results using several approaches that require micro-level data.<sup>10</sup> These data show similar upward movement in the dynamics of the total household wealth-to-income ratio as in the aggregate accounts of the United States (FA and NIPA), although the magnitudes are somewhat smaller for the latter.

The aggregate wealth-to-income ratios in 1989 and 2019 we use for our decompositions are calculated from the SCF as the ratio of the sum of the wealth of each household to the sum of incomes of each household, respectively denoted  $\left(\frac{w}{y}\right)_{89}$  and  $\left(\frac{w}{y}\right)_{19}$ . In our baseline, we include all household wealth either directly reported in or constructed from the SCF (including estimates of DB pensions from Sabelhaus and Volz (2020)) in our measure of wealth.<sup>11</sup> To explore robustness, we also provide calculations where we exclude wealth in a primary residence from the baseline measure of wealth.<sup>12</sup> Our measure of income is the total of components available in SCF, and does not vary with the definition of wealth used either in the baseline or the robustness scenarios.<sup>13</sup>

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<sup>8</sup>SCF only directly measures pensions in defined contribution plans. Defined benefit pension entitlements calculated by Sabelhaus and Volz (2020) represent their termination value, which is the legal obligation of employer plans, and corresponds to the measure of defined benefit pension entitlements (both funded and unfunded) in Financial Accounts. We thank the authors for sharing their estimates with us.

<sup>9</sup>Sabelhaus and Volz (2020) also provide estimates of social security wealth using SCF, in addition to defined benefit pension wealth, using both termination and expected values of such wealth. However, similar to Auclert et al. (2021) conceptually we think of social security wealth as a future transfer, and do not include it in our measure of household wealth.

<sup>10</sup>Later in the section we also report results from micro data scaled to Financial Accounts and National Income and Product Account (NIPA) aggregates.

<sup>11</sup>In particular, we do not exclude vehicles as a measure of consumer durables from household wealth in the SCF. Consistent with this approach, our measure of saving rates in the next section also includes consumer durables, as the FA concept of saving rate, but unlike the NIPA measure of saving rate. However, the difference in saving rates implied by inclusion of vehicles is very small. There is some difference associated with using NIPA or FA saving, as the FA saving is more noisy. However, the dynamics over time of these saving rates are quite similar.

<sup>12</sup>Our preferred measure of wealth includes housing. An additional reason for this has to do with the subsequent analysis of group-wise changes in wealth and saving rates in Section 2.1. We construct saving rates by components of household wealth following the approach in Mian et al. (2021b). Thus, while it is also possible to exclude the components of housing wealth – both in assets and liabilities – from the construction of saving rates, the relevance of this measure in comparison to other studies using standard measures of saving from the data is less clear. Saving in housing expressed as the net new housing also represents a non-trivial component of saving.

<sup>13</sup>Following Fagereng et al. (2019) and Eika et al. (2020) we have also examined the case where we include in the SCF definition of income a measure of imputed housing rents of homeowners, constructed by distributing NIPA reported rents according to the value of housing of SCF respondents. When applying this

The aggregate wealth-to-income ratio in the SCF increased from 5.61 in 1989 to 8.43 in 2019, which is an increase of about 2.82, or close to 50%. This is the increase associated with an inclusive wealth measure from the SCF. When we exclude net housing wealth from this measure, the increase in the ratio is of similar magnitude at 2.65. The increases are all substantial relative to 1989 levels.<sup>14</sup> To examine the within versus between components of increased wealth holdings, we apply a simple shift-share methodology in the main text, and report robustness results using a regression based decomposition in Appendices B.2 and ??.

For the shift share analysis, we place households in  $l$  bins, with  $N_i$  households in a bin  $i = 1, \dots, l$ . The change in the aggregate wealth-to-income ratio can be decomposed as follows:

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \underbrace{\sum_i \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89} \left[\left(\frac{y_i}{y}\right)_{19} - \left(\frac{y_i}{y}\right)_{89}\right]}_{\text{between group or compositional effect}} + \underbrace{\sum_i \left(\frac{y_i}{y}\right)_{19} \left[\left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{19} - \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89}\right]}_{\text{within group}}, \quad (1)$$

where the first summation term represents the between group component, using 1989 as the base year for income and wealth profiles, and the second one represents the within group component.<sup>15</sup> In this expression,  $y_i$  is the total income in bin  $i$ ,  $\bar{y}_i$  is the average income in bin  $i$ ,  $\bar{w}_i$  is the average wealth in bin  $i$  and finally  $y$  is the total income across all bins. All nominal variables are converted into real variables indexed in 2019 dollars. As can be seen from Equation (1), the changes in the total wealth-to-income ratio can be divided into the between group component determined by the shift in the share of income going to each of the individual groups ( $y_i/y$ ) and the within group component determined by changes in the (average) wealth-to-income ratio of each group ( $\bar{w}_i/\bar{y}_i$ ). If the wealth-to-income ratios of individual groups were stable across time (e.g.,  $(\bar{w}_i/\bar{y}_i)_{19} = (\bar{w}_i/\bar{y}_i)_{89}$  for all groups  $i$ ), the change in the aggregate wealth-to-income ratio would need to be fully explained by the between group component (i.e., by the change in income shares alone). However, at the other extreme, if the income and age distributions remained stable across time (e.g.,

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definition of income with the baseline measure of wealth, we find that the contribution of the within-group component to the overall change in the wealth-to-income ratio is largely unchanged. For this reason, we keep the original measure of income from SCF in our baseline results.

<sup>14</sup>In the scaled wealth and income data, the ratio changes by 171pp from 4.27 in 1989 to 5.98 in 2019. The literature has also used other definitions of wealth ratios, for example, normalized by GDP. While the exact changes in these ratios may depend on what goes into their numerator/denominator, they all have increased substantially over time.

<sup>15</sup>In Appendix ??, we clarify the close relationship between the shift-share and the regression based decomposition. This discussion helps to highlight under what conditions the first component represents the between-group component and the second component represents a within-group component.



if  $(y_i/y)_{19} = (y_i/y)_{89}$  for all groups  $i$ ), then the within group components would need to account for all the change in the aggregate wealth-to-income ratio.<sup>16</sup>

We start by dividing the population households into age groups, defined by the age of the head of the household, to look narrowly at the effects of demographic changes in isolation. Then, we divide the population of households into income groups to examine only the effects of changes in the current income distribution. Finally, in our preferred specification, we combine the two and place households into age-income specific bins.<sup>17</sup> The results of the shift share analysis for these different groupings are presented in Tables 1 and 2.

Table 1  
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio  
Between 1989 and 2019

Groups	Total Change	Between	Within	Fraction due to Within (%)
Age Groups				
5 age groups	2.819	0.944	1.875	66.5
12 age groups	2.819	0.984	1.835	65.1
Income Groups				
6 income groups	2.819	0.175	2.644	93.8
12 income groups	2.819	0.179	2.640	93.6

Note: The 5 age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the 12 age groups are: <25, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, and 75+. The 6 income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+ and the 12 income groups: 0-20, 20-40, 40-60, 60-80, 80-120, 120-160, 200-250, 250-500, 500-750, 750-1250, 1250+. Wealth includes defined benefit pensions.

<sup>16</sup>Work by [Auclert et al. \(2021\)](#) is the closest to this paper in terms of quantifying the contribution of population aging, i.e. between-group component with 5-year age groups in place of  $i$ , to the change in the wealth-to-income ratio in the US between 1950 and 2016.

<sup>17</sup>In their shift-share analysis of the changes in saving to national income ratio, [Mian et al. \(2021b\)](#) test the relative importance of aging versus income inequality drivers over the 1950-2019 period. For this reason, they choose to focus separately on age groups and within-birth-cohort income distribution groups defined by 10th, 50th, and 90th income percentiles. [Feiveson and Sabelhaus \(2019\)](#) also look at within-birth-cohort permanent income groups which are only available for the 1995-2019 period. When using normal income for the formation of income groups and income measure itself, we find that over the period between 1995 and 2019, the within-group component is responsible for 55% of the change in our benchmark measure of the wealth-to-income ratio, including defined benefit plans.

In Table 1, we report results for the more narrow focus on either only age or income groups.<sup>18</sup> With respect to the results based on demographics the table presents two breakdowns: one based on 5 age groups and one based on 12 age groups. For these two breakdowns, we get very similar results: the within component explains about 65 percent of the change in the wealth-to-income ratio.<sup>19</sup> Then, we look at two groupings based only income: one based on 6 groups and one based on 12 groups. In both of these cases, the between component only explains about 7 percent of the change, leaving 93 percent of the change to the within-group component.<sup>20</sup>

In Table 2, we present results for our preferred approach, where we allow for 30 groups as the product of 5 age groups and 6 income groups. These results use two different measures of wealth: our baseline measure inclusive of all wealth and the baseline measure less net housing wealth (primary residence).<sup>21</sup> For comparison between the survey and aggregate data, we also report in Table 2 the results of the shift-share analysis when rescaling SCF estimates of wealth and income to match the FA and NIPA aggregates ("scaled" estimates). The latter approach is used in the literature, such as [Feiveson and Sabelhaus \(2019\)](#), [Mian et al. \(2020\)](#), and [Bauluz and Meyer \(2019\)](#). It builds each group's wealth using its shares of different assets and liability classes in SCF and values of their counterpart FA classes.<sup>22</sup> The same is done on the income side where SCF reports income from different sources, which are matched to their corresponding aggregates in NIPA.<sup>23</sup> As shown in Panels A and

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<sup>18</sup>Appendix B.1 also presents the results of the decomposition of the changes in wealth-to-income ratio between 1989 and 2019 into within and between-group components with additional income groups at the higher end of the income distribution. The results using additional groups are similar to the benchmark 6 and 12 income groups.

<sup>19</sup>The compositional age effect in [Auclert et al. \(2021\)](#) computed for the 1950-2016 period and using 2016 for base profiles of labor earnings and wealth is responsible for 105 out of 118 percentage points increase in the wealth-to-GDP ratio. Over the period studied in this paper the compositional effect in [Auclert et al. \(2021\)](#) is about half of that in the full period, while the actual change in wealth-to-GDP ratio is similar between 1989 and 2016 and 1950 and 2016. In what follows, we also discuss the results of using 2019 as the base year for the between-group component calculation, which is closer to 2016 used in [Auclert et al. \(2021\)](#).

<sup>20</sup>As mentioned earlier, this definition of income groups differs from that used in [Mian et al. \(2021b\)](#) analysis of the changes in aggregate saving rate, which also controls for life-cycle effects by looking at top, middle and bottom parts of income distribution within each cohort. We similarly condition on age differences, when looking at the product of income and age groups in what follows.

<sup>21</sup>In the remainder of the sections pertaining to empirical analysis, we would refer to SCF data plus DB pensions as simply raw SCF data, for ease of the exposition.

<sup>22</sup>We use the aggregates as reported directly in FA, as opposed to combining aggregates for some asset classes from SCF and others from FA. But the results do not substantially change when we use a combination approach instead.

<sup>23</sup>In the benchmark, we use the definition of gross NIPA income less imputations for owner-occupied housing rents. However, we have also used other measures of income, including with imputed owner-occupied rents, and the results using these other measures are similar.

B, the two sets of results are quite similar. The within component — that is, the component associated with changes in the wealth-to-income ratio of different groups — accounts for between 57 and 65 percent of the change with the between component explaining around 40 percent.

The results in Table 2 are obtained using 1989 as the base year for each group  $i$ 's initial profiles. In Appendix B.1 we also check the robustness of these results when changing the base year to 2019. When doing so, as suggested by Mian et al. (2021b), the importance of the between component increases, helping to explain some of the difference in our results relative to those reported in Auclert et al. (2021). However, even with the change in the base year the within-group component accounts for more than 50% of the change in the aggregate wealth-to-income ratio between 1989 and 2019.<sup>24</sup>

Table 2  
Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Shift Share Decomposition

Definition	Total Change	Within (%)	Between (%)
Panel A: Raw SCF Data			
Wealth (baseline)	2.819	61.6	38.4
Wealth less housing	2.649	61.4	38.6
Panel B: Scaled SCF Data			
Wealth (baseline)	1.71	57.4	42.6
Wealth less housing	1.64	65.9	34.1

Note: The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+.

<sup>24</sup>It is worth noting that the sensitivity of results to base year choice tends to decrease as we increase the number of groups. However, it must also be recognized that some of the groups start to have rather few observations when we go above 30, explaining the choice of the number of groups for our baseline results. Nonetheless we did explore how our decomposition results change when we considerably increase the number of groups. For example, when we allow for 75 groups (15 income groups and 5 age groups), our within group estimate declines to around 48% when using 1989 as the base year, and this does not change much if we change the base year to 2019. Given this, we are comfortable interpreting our results as suggesting that both the within and between components are close to equally important in explaining the increase in wealth to income ratios.

It must be immediately noted that these decomposition results — by themselves — do not imply that within group *desired* wealth holdings have necessarily gone up. Instead, if households' wealth holdings are sticky, it could be that these high levels of within-group increases in wealth holdings simply reflect the fact that falls in real interest rates have led to increased valuation of wealth, and that households in 2019 are holding much more wealth than they desire relative to similar households in 1989. This could be the case if households face constraints on adjusting their portfolios. This is especially likely for housing, which is why we also reported results excluding housing. As we saw, the results are not driven by housing wealth. Nonetheless, to explore the possibility that households hold wealth above their target level more thoroughly, we need to examine the changes in saving rates by age-income groups. We do so focusing on total saving rates.

## **2.1 Within-group saving behavior: Are households in 2019 trying to shed their increased wealth?**

In the previous section we documented that a large share of the increase in the aggregate wealth-to-income ratio in the US over the period 1989-2019 is accounted for by increases in wealth for given levels of age and income, that is, it is predominantly a within group phenomenon. There are at least two potential interpretations of such an observation. On the one hand, increases in wealth-to-income ratio could reflect increases in desired wealth holdings due to low expected returns on assets, higher longevity or decreased economic growth. On the other hand, such increases in wealth could reflect unanticipated valuation effects, where the observed higher wealth holdings reflect wealth holdings that are above their desired levels. To help discriminate between these two sets of possibilities, in this section we look at the changes in within group saving patterns over the same period. In particular, if the observed within group increases in wealth-to-income ratios reflect wealth levels in 2019 that are above desired levels, then we should see household groups with large increases in wealth wanting to spend more and save less to get their wealth back down to its target level. Accordingly, we should see them decrease their savings rates.<sup>25</sup> Hence, the absence of a negative relationship between increased wealth and changes savings rates would indicate that the extra wealth holdings are likely desired not excessive.

In line with the previous section, we focus on within group changes in saving rates for

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<sup>25</sup>Fagereng et al. (2019) ask a similar question whether households who experience capital gains sell off the assets subject to price increases to consume. They find evidence against such behavior and show that it is consistent with a model where asset price increases are driven by declining asset returns, as opposed to growing dividends.

the 30 groups we used for our analysis of changes in wealth-to-income ratios. We measure saving in the SCF using synthetic saving approach, widely adopted in the literature, which approximates saving by each group by netting out valuation effects from changes in their wealth between two SCF waves.<sup>26</sup> Our saving rates are calculated over a three-year window. Saving rates for 1989-92 and 2016-2019 periods, respectively, correspond to the start and the end of our 30-year period used to analyze changes in the aggregate wealth-to-income ratio in the US. In our robustness exercises using the SCF, we also exclude net inheritances from changes in wealth, which does not materially change the results.<sup>27</sup>

We follow the approach of the previous section in using both unscaled/raw SCF data, as well as scaled to the aggregates SCF data to construct group savings rates and their changes. For valuation effects we apply asset/debt inflation factors from Mian et al. (2020), which are aggregate in nature and are available until 2016 inclusive, and use their methodology to extend them to 2019. Appendix A provides further details of the saving rate construction.

Table 3  
Correlation between Group Changes in Wealth-to-Income Ratios and Changes in Saving Rates: Raw and Scaled SCF Data, 30 Age-Income Groups

	Raw SCF	Scaled SCF
$\text{corr}(\Delta(s/y), \Delta(w/y))$	-0.05	0.16

Note: Correlation is computed using 30 age-income groups constructed using SCF data as defined previously.

Table 3 presents correlations between changes in wealth-to-income ratios and changes in saving rates using these two different approaches. Using raw SCF data to compute the correlation between group changes in wealth-to-income ratios and saving rates results in a coefficient of -0.05, and with scaled SCF data it is 0.16. Both of these numbers suggest that groups that faced greater increases in wealth-to-income ratios do not appear to systematically reverse this accumulation by decreasing their saving rates.<sup>28</sup> In Figure 3,

<sup>26</sup>For other papers using synthetic saving approach see Mian et al. (2021b) and references therein. This approach of decomposing total changes in wealth into the component associated with capital gains and non-capital gains component is also used in the FA approach to calculating saving, with the latter conceptually corresponding to the measure of saving in NIPA.

<sup>27</sup>Accounting for inheritances has a zero net effect in the aggregate, as inheritances received should equal inheritances bequeathed, but within groups these inflows and outflows may not be equal, potentially affecting group-wise changes in saving rates.

<sup>28</sup>Amongst our 30 benchmark groups, we find that all of the groups in the top income grouping, except

we complement the evidence on correlations from Table 3 by plotting the changes in saving rates against the changes in log wealth for all the groups that experienced increases in wealth. Given that the saving rates constructed using raw SCF data were low relative to aggregate measures of saving rates in FA/NIPA, for this figure we are using results based on the scaled SCF measures of saving rates. The average change in savings rates for this subset is slightly positive. Moreover, as can be seen in the figure (and is confirmed by the correlation), higher increases in wealth are not on average associated with larger decreases in saving rates. It must be recognized that our measure of saving rates, which is common to the literature, is quite noisy. Accordingly, we witness substantial variation in saving rates. Nonetheless, we view these patterns as providing support to the notion that increases in within group wealth-to-income ratios documented in the previous section are more likely reflecting changes in desired wealth holdings as opposed to reflecting wealth holdings that exceed desired levels.

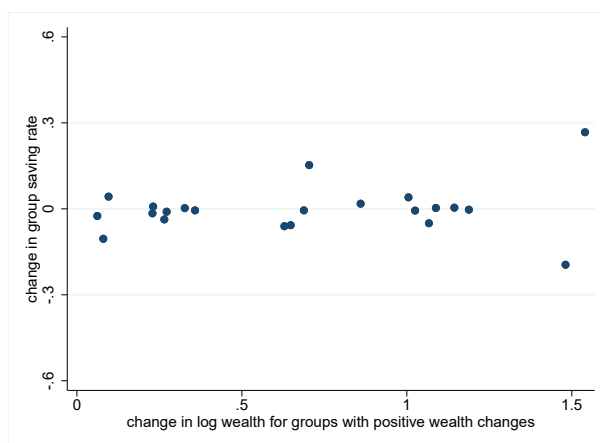


Figure 3

Change in saving rates vs. change in log wealth for age-income groups with wealth increases between 1989 and 2019

Sources: Survey of Consumer Finances values scaled using aggregates from the Financial Flow Accounts and National Income and Product Accounts.

## 2.2 Focusing on the 55-64 age cohort

In this section, we focus on the change in wealth for the 55-64 age group, using our baseline measure inclusive of defined benefit pensions. Since this is mainly a pre-retirement one, did not decrease their saving rates, which is consistent with findings in [Mian et al. \(2021b\)](#) using averages for 1963-1982 and 1995-2019 periods and the top 10% of the within-cohort income distribution. However, the time periods and the income group definitions in the two studies are not fully comparable.

group, its asset holding most likely reflect a combination of life-long saving decisions which had to balance different forces, including retirement needs. We start by considering the relationship between income and wealth in each of the SCF survey years, focusing on the range of real incomes between 40 000\$ to 400 000\$, as it covers 94% of income and avoids outliers.<sup>29</sup> More specifically, we run a regression of wealth on income and income squared separately for 1989 and 2019.

In Figure 4, we provide the scatterplot of the raw data on household level wealth against household income for both 1989 (green circles) and 2019 (grey circles), all in 2019 dollars. We also superimpose a quadratic regression line of the relationship for each of the two years. These are depicted in the figure in solid green and grey lines respectively. The coefficients for the regression are reported in Table 4. As can be seen in the figure, and confirmed in the table, the relationship between wealth and income is approximately linear in both 1989 and 2019. The relations between wealth and income changed considerably from 1989 to 2019. In 1989, each additional dollar of income was associated with an increase of 9.2\$ of wealth. By 2019, each additional dollar of income was now associated with additional 14.3\$ of wealth.<sup>30</sup> A goal of our later quantitative exercise will be to decompose forces that may have lead to such large changes. These will include changes in longevity, changes and growth and changes in real interest rates

Table 4  
Regression of wealth on income: linear and quadratic fit (1989 and 2019)

	Income (t-stat)	Income <sup>2</sup> (t-stat)	intercept
1989	9.22 (15.3)	–	-27 092
1989	9.83 (5.6)	-1.8*10 <sup>-6</sup> (.29)	-61 952
2019	14.3 (24.5)	–	-370 409
2019	14.1 (7.9)	7.7*10 <sup>-7</sup> (.14)	-353 925

<sup>29</sup>The results also look similar whether we consider the 20 to 400 thousand dollars income range. We choose to exclude the 0 to 20 thousand dollars group, as the share of the 55-64 year old population in this range experienced a non-negligible decline between 1989 and 2019. Another reason for focusing on the income range chosen is to deal with early retirements which could more likely show up in the very bottom income range. However, we also conduct robustness analysis to deal with this issue explicitly by using information from the SCF.

<sup>30</sup>In the Appendix, for robustness we also look at the measure of wealth excluding primary real estate wealth. We also consider other age groups around the 55-64 age group to see if they display similar patterns in their wealth to income relationships.

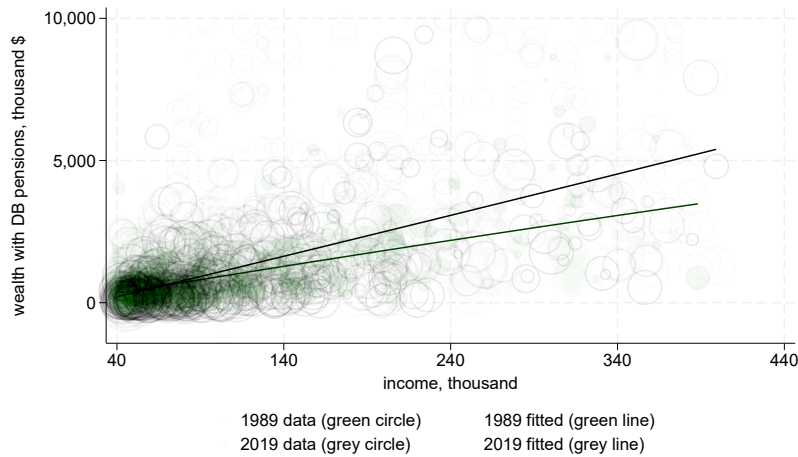


Figure 4  
Relation between wealth and income: 1989 and 2019

Sources: Survey of Consumer Finances, 1989 and 2019. This figure plots the raw observations on wealth and income from SCF and the fitted lines from the quadratic regression of wealth on income, separately for 1989 and 2019.

Although Figure 4 indicates that for most income levels in the 55-64 year old age group, wealth increased substantially between 1989 and 2019, it is important to note that the regression line actually pivots around an income level of approximately 65 000\$. For households with income below this level, average wealth actually fell between the two periods. This implies that wealth-to-income ratios actually fell for low income groups, while they rose for higher income groups. This can be seen in Figure 5, where we now plot wealth-to-income against income for both 1989 and 2019. We again include fitted quadratic lines of the relationship in each year. On this figure, the fall in wealth for lower groups becomes clearer.<sup>31</sup> We can also see that the wealth-to-income ratio in 1989 was around 9 across different income groups, but for income groups above 65 000\$ the wealth to income ratio increased to around 14 in 2019.

Given these changes in wealth-to-income, we can again use the shift share approach to evaluate the fractions of overall wealth-to-income for the group that is due to a change in the distribution of income versus a change in the level of wealth holding income constant. Recall from Equation 1, that the within component was calculated as the difference between  $\sum_i \left(\frac{y_i}{y}\right)_{19} \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{19}$  and  $\sum_i \left(\frac{y_i}{y}\right)_{19} \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89}$ . These two quantities were equal to 8.5 in 1989

<sup>31</sup>While increased longevity could have contributed to an increased desire to accumulate wealth for higher income groups, this factor is unlikely to explain much of the changed behavior of the lower income groups as their life expectancy has not changed much over this period.



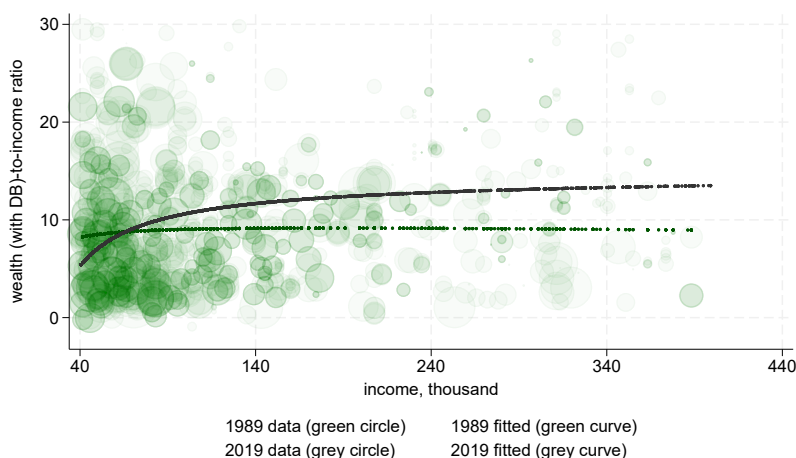


Figure 5

Wealth-to-income ratios constructed using raw SCF data and prediction from a quadratic regression of wealth on income for 1989 and 2019

Sources: Survey of Consumer Finances, 1989 and 2019.

and 11.3 in 2019.

### 2.3 Deriving the Asset-demand Interest-rate Relationship: when inter-temporal substitution competes with retirement motives.

In the previous section we documented that for households, within the same age and real income group, their desired asset holdings appears to have increased substantially over the 1989-2019 period. In this section, we propose a tractable framework aimed at capturing different potential forces that could help explain this observed change. Our approach builds on a model similar to that of [Gertler \(1999\)](#) that integrates both inter-temporal substitution forces and retirement preoccupations in wealth accumulation.<sup>32</sup> In particular, these two forces will be shown to interact in a manner that gives rise to rich specification of asset demand. A key property we will highlight in this section is why the effects of interest rates on long run asset demands is likely to be C-shaped.

<sup>32</sup>We depart from [Gertler \(1999\)](#) by maintaining the more common CRRA utility specification instead of adopting RINSE preferences. [Carvalho et al. \(2016\)](#) uses the model of Gertler to examine equilibrium rates. More recently, [Galí \(2021\)](#) introduces retirement in a New Keynesian model with logarithmic utility in which there are multiple steady state real rates that are related to the size of bubbles. In a two-period OLG model with nominal rigidities, [Plantin \(2022\)](#) also examines the case where a Taylor rule may create monetary bubbles. We do not explore the possibility of bubbles in our analysis.

## 2.4 The Household Wealth accumulation problem

When thinking about consumption and wealth accumulation decisions, it is common to think about people in different states. As is standard in simple OLG models, we can think of a household in one of three states: an active work state, a retirement state and a death state. Following [Blanchard \(1985\)](#), [Yaari \(1965\)](#) and [Gertler \(1999\)](#) we want to think of these states as evolving stochastically. To be more precise, let us assume that a person starts life in a work state and transits out with instantaneous probability  $\delta_1$ . In the absence of fixed retirement dates, this shock can be thought as a health shock. At this transition, with probability  $q$ , the person retires and with probability  $(1 - q)$ , the health shock is severe, and the person dies. If the person retires, the person will die with instantaneous probability  $\delta_2 \geq \delta_1$ . If we denote the expected discounted utility of entering the retirement state at time  $t$  by  $V_t$ , we can express the utility of an active household, that is a household in the work state, as:

$$\int_0^{\infty} e^{-(\delta_1 + \rho)t} \left[ \frac{c_t^{1-\sigma}}{1-\sigma} + \delta_1 q V_t \right] dt, \quad \sigma > 0$$

where  $c_t$  is consumption,  $\rho$  is the subjective discount rate and  $\sigma > 0$  is the inverse of the elasticity of substitution ( $1/\sigma$ ), or alternatively the risk aversion parameter.

**The retiree's decision problem.** For the household in the retirement state, the preferences are given by:

$$\int_0^{\infty} e^{-(\delta_2 + \rho)\tau} \frac{c_\tau^{1-\sigma}}{1-\sigma} d\tau, \quad \sigma > 0$$

The budget constraint facing the retired household is given by:

$$\dot{a}_t = a_t r_t + p b_t - c_t,$$

where  $a_t$  is the asset holding of a retired person at time  $t$ ,  $r_t$  is the return on the asset  $a$  and  $p b_t$  is a pension benefit. As can be seen from the budget constraint of the retirees, moving into the retirement state is associated with the replacement of labor income by a pension benefit. As long as the pension benefits is lower then pre-retirement labour income, there will tend to be an incentive for retirement savings. For ease of presentation, we will begin by solving the household problem in the absence of any pension benefits ( $p b_t = 0$ ). In Section xx, we will reintroduce pension benefits and show how the analysis extends easily to cover this more general case.

Given our structure, the discounted expected utility of a household who retires at time  $t_1$ ,  $V_{t_1}$ , can be solved explicitly and expressed as<sup>33</sup>

$$V_{t_1} = \frac{(a_{t_1})^{1-\sigma}}{1-\sigma} \left[ \int_{t_1}^{\infty} e^{-\int_{t_1}^t \frac{1}{\sigma} [(\rho+\delta_2)-(1-\sigma)r(\tau)] d\tau} dt \right]^{\sigma},$$

where  $a_{t_1}$  is the level of assets held by the household at time of retirement. For convenience, we will also express  $V_{t_1}$  as

$$V_{t_1} = V(a_{t_1}, \Gamma_{t_1}) = \frac{(a_{t_1})^{1-\sigma}}{1-\sigma} [\Gamma_{t_1}]^{\sigma},$$

where

$$\Gamma_{t_1} = \int_{t_1}^{\infty} e^{-\int_{t_1}^t \frac{1}{\sigma} [(\rho+\delta_2)-(1-\sigma)r(\tau)] d\tau} dt,$$

with  $\Gamma_{t_1}$  being a function of the whole future path of returns  $\{r_t\}_{t_1}^{\infty}$ . Expressing utility as  $V(a_{t_1}, \Gamma_{t_1}) = \frac{a_{t_1}^{1-\sigma}}{1-\sigma} [\Gamma_{t_1}]^{\sigma}$  makes it clear that the utility of someone who retires at time  $t_1$  depends on both the total asset at the time of retiring and the entire path of asset returns over the retirement period. As we shall see, the degree of inter-temporal substitution  $\frac{1}{\sigma}$  will play an important role in controlling how asset returns affect marginal value of assets.

For future reference, it is useful to note that  $\Gamma_{t_1}$  obeys the following differential equation

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r_t \right]. \quad (2)$$

To see most easily how asset returns affect retirement utility, note that if the return on asset is constant,  $r_t = r$ , then  $V_{t_1}$  can be expressed as

$$V_{t_1} = \frac{(a_{t_1})^{1-\sigma}}{1-\sigma} \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r \right]^{-\sigma} \quad (3)$$

We can see from 3 that higher  $r$  increases utility in both the case where  $\sigma < 1$  or when  $\sigma > 1$ , that is, retired individuals like higher interest rates as this gives them a superior income stream. However, what will play an important role in our analysis is how higher  $r$  affects the marginal value of  $a_{t_1}$  to a retiree. This is given by the following key lemma.

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<sup>33</sup>The expected utility associated with the retirement state is found by first solving for the optimal consumption path, which is governed by the Euler equation  $\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_2}{\sigma}$  and then integrating the implied utility flow over the expected duration of retirement.

**Lemma 1.** For a fixed  $r$ , the marginal value of assets to a retiree is decreasing in  $r$  when  $\sigma > 1$  and is increasing in  $r$  when  $\sigma < 1$  since  $\frac{\partial^2 V_{t_1}}{\partial a_{t_1} \partial r} = a_{t_1}^{-\sigma} (1 - \sigma) \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right]^{-\sigma - 1}$ .

In general, the effect of asset returns on the marginal value of assets for retirees depends on  $\sigma$ . As noted in Lemma 1,<sup>34</sup> this marginal value is decreasing in  $r$  when  $\sigma > 1$ . In other words, when a retiree has limited opportunities to inter-temporally substitute consumption across time, the retiree will view assets at time of retirement to have a greater marginal value when interest rates are low than when they are high. <sup>35 3637</sup>

Another comparative static one obtains from 3 is non surprising result that the marginal value of assets is decreasing in  $\delta_2$ , that is, higher expected longevity in retirement increases the marginal value of assets. Both these property of the marginal value of assets will be important drivers of the wealth accumulation decisions of active households.

**The active household's decision problem.** Let us now turn to the decision problem of an active household. Its decision problem will incorporate the continuation value of assets in retirement and can be written as:

$$\int_0^{\infty} e^{-(\delta_1 + \rho)t} \left[ \frac{c_t^{1-\sigma}}{1-\sigma} + \delta_1 qV(a_t, \Gamma_t) \right] dt,$$

subject to

$$\dot{a}_t = y_t - c_t \tag{4}$$

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<sup>34</sup>Lemma 1 can be trivially extended to include the case of log preferences. In this case, the marginal value of assets is independent of interest rates, i.e.,  $\frac{\partial^2 V_{t_1}}{\partial a_{t_1} \partial r} = 0$ .

<sup>35</sup>When  $\sigma > 1$ , a rise in interest rates causes the optimal path of post-retirement consumption to be higher at all dates and hence the marginal value of assets is lower. This is easily understood and intuitive. In contrast, when  $\sigma < 1$  different interest rates cause optimal paths of post-retirement consumption to cross; with retirees consuming initially less in a higher interest rates environment but having their consumption decline more slowly over time. Because of this crossing property, the effect of interest rates on the marginal value of assets is not straightforward when  $\sigma < 1$ . Lemma 1 indicates that the net effect is that higher interest rates increase the marginal value of assets when  $\sigma < 1$  due to this crossing feature.

<sup>36</sup>It is worth noting that, although we have not explicitly introduced an annuity market for transforming asset wealth  $a_t$  into a guaranteed income stream, Lemma 1 is not dependent on the presence or not of such a market. The content of Lemma 1 would remain identical if we were to allow for an annuity market similar to that in Blanchard (1985).

<sup>37</sup>Like Gertler (1999), a key assumption is the absence of a pension system which acts as a perfect insurance market against loss of labor income. The absence of such market implies consumption in retirement relies at least in part on the accumulated savings when active.

with  $y_t = w_t + r_t a_t - T_t$ , where  $y_t$  is disposable income,  $w_t$  is labor income and  $T_t$  are taxes.

The consumption Euler equation for the active household is

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma} \delta_1 q V_a(a_t, \Gamma_t) \quad (5)$$

Relative to a standard infinitely lived agent Euler equation, this Euler equation incorporates forces associated with both inter-temporal substitution and retirement preoccupations as in [Gertler \(1999\)](#) and [Grandmont \(1985\)](#). The first term in this Euler equation maintains the standard substitution effect of interest rates on consumption. However, this effect now relates to short-term interest rates movements holding the future path of interest rates constant. When both short-term and long-run interest rates move together the net effect is more involved. The additional term in the Euler equation —  $\frac{c_t^\sigma}{\sigma} \delta_1 q V_a(a_t, \Gamma_t)$  — represents the incentive to save due to retirement motives and thus is affected by future interest rates. Given this term is always positive, it implies that retirement adds a force towards postponing consumption and favoring asset accumulation.<sup>38</sup> The key element for us is that the retirement incentive to save is affected by long run returns to savings. In particular, when interest rates are constant,  $r_t = r$ , we have seen that  $V_{a,r}(a_t) < 0$  when  $\sigma > 1$ . Hence, interest rates have two opposing effects in our set-up when  $\sigma > 1$ . Low interest rates will favor higher consumption today due to inter-temporal substitution forces, while at the same time, low interest rates are an incentive for greater retirement savings if the low interest rates are viewed as persistent.

To help further highlight implications of this Euler equation, it is helpful to examine the implied long-run asset holdings of the active household when the return of asset is constant and therefore  $\Gamma_t = [\frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r]^{-1}$ . We will denote an active household's steady state asset holding by  $a^{a,ss}$ . [Proposition 1](#) indicates that  $a^{a,ss}$  is attractive and describes the key properties of the steady state asset-to-income ratio  $\frac{a^{a,ss}}{y}$ .

**Proposition 1.** *With a fixed  $r$ , the asset-to-income ratio of active households will converge to*<sup>39</sup>

$$\frac{a^{a,ss}}{y} = (\delta_1 q)^{\frac{1}{\sigma}} \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma}}, \quad (6)$$

<sup>38</sup>This force is also present in models with warm-glow bequest motives, but in that case it does not depend on interest rates, which is the key feature for our purposes.

<sup>39</sup>When  $\sigma = 1$ , this equation implicitly defines the asset-to-income ratio as a function of interest rates.

when  $r$  is in the interval defined by  $[\frac{\rho+\delta_1-r}{\delta_1 q} (\frac{\rho+\delta_2}{\sigma} - \frac{(1-\sigma)r}{\sigma})^\sigma]^\frac{1}{\sigma} > \max[0, r]$ .<sup>40</sup>

This steady state asset-to-income ratio of active households is increasing in longevity (i.e. decreasing in  $\delta_2$ ) and decreasing in  $\rho$ . Moreover, if  $\sigma \leq 1$ , then  $\frac{a^{a,ss}}{y}$  are monotonically increasing in asset return  $r$ , while if  $\sigma > 1$ , they are C-shaped in  $r$ .

See Appendix C.1 for the proof.

The first two property noted in Proposition 1 are straightforward and intuitive. The more insightful element of Proposition 1 relates to the effects of  $r$  on desired long-run asset holdings. In particular, we see that if  $\sigma \leq 1$ , then desired long run asset holdings would be monotonically increasing in  $r$  because the substitution effect always dominates retirement savings effect. In contrast, when  $\sigma > 1$  the effects of  $r$  on long-run asset holdings are non-monotonic. For high levels of returns, desired holdings are increasing in  $r$ , while for low returns they are decreasing in  $r$ . To understand this effect, recall that interest rates have two effects in this model. At low interest rates, households are encouraged to consume more, and accumulate less, through the standard inter-temporal substitution channel. However, retirement preoccupations play a counterbalancing role. When long-term interest rates are low and  $\sigma > 1$ , active households have an increased marginal incentive to accumulate assets for retirement needs. What Proposition 1 indicates is that there will be a point of reversal of the effect of steady state  $r$  on accumulation incentives. When  $r$  is sufficiently high, a marginal increase in steady state  $r$  would lead to more accumulation as the positive substitution effect dominates the decreased retirement need effect even if  $\sigma > 1$ . When interest rates are low, the marginal value of asset becomes very high. This causes the need for retirement wealth to dominate the inter-temporal substitution effect and gives rise to the C-shape asset demand.<sup>41</sup>

The shape of the active household's steady state asset-to-income ratio  $\frac{a^{a,ss}}{y}$  is illustrated in Figure 6 for the case when  $\sigma > 1$ . The figure illustrates the C-shape asset demand function. Moreover, we can see that the asset-to-income ratio (when  $\sigma > 1$ ) is delimited by two levels of  $r$ . As  $r$  tends to  $\rho + \delta_1$  from below, the steady state asset-to-income of active households will tend to infinity. As  $r$  tends to  $\frac{\rho+\delta_2}{1-\sigma} < 0$  from above,  $\frac{a^{a,ss}}{y}$  will tend

<sup>40</sup>If  $r$  is not in the interval, asset holdings do not converge.

<sup>41</sup>Our paper has some similarities with the work of Brunnermeier and Koby (2019) on the reversal interest rate. In their work, there is a reversal rate of interest whereby interest rates below the reversal rate become contractionary. Their reversal rate result comes from banking frictions. Our set-up can also be thought as having a reversal rate, which we denote  $\bar{r}$ . Our reversal rate arises from expected income effects in retirement that drive up households' desired savings while working and therefore depress consumption.

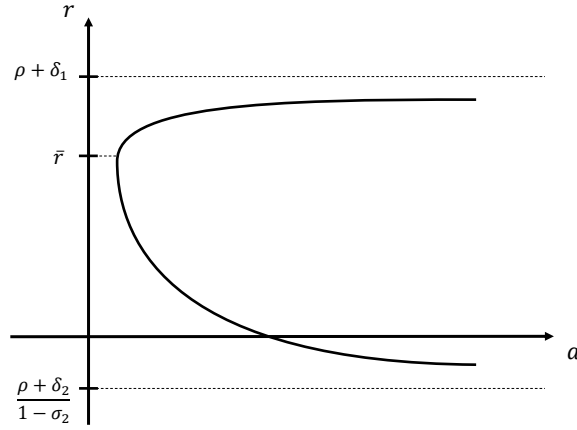


Figure 6  
Active households' long-run asset demand

again toward infinity. When  $\sigma > 1$ , there exists also a threshold or point of inflexion

$$\bar{r} = \left[ \frac{\sigma(\sigma - 1)(\rho + \delta_1) - (\rho + \delta_2)}{(\sigma - 1)(\sigma + 1)} \right],$$

such that the asset-to-income ratio demand of active households is increasing in interest rates when  $r$  is above  $\bar{r}$  and is decreasing in interest rates when  $r$  is below  $\bar{r}$ .

Up to now, we have not allowed for growth in labour income. Extending this model for growing labour income – when seen as due to an aggregate growth trend– is straightforward. In fact, the household problem then inherits balanced growth properties, with the system converging to constant asset-to-income growth path. In particular, if income grows that the instantaneous rate  $g$ , then it is easy to verify that optimal consumption decision with generate a steady state asset-to-income ratio given by:

$$\frac{a^{a,ss}}{y} = (\delta_1 q)^{\frac{1}{\sigma}} \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right]^{-1} [\rho + \delta_1 + \sigma g - r]^{\frac{-1}{\sigma}}, \quad (7)$$

Equation 7 will form the basis for examining what force may best help explain the observed change in asset-to income ratios. This will include examining the potential role of increased longevity (decreased  $\delta_2$ ), fall in economic growth  $g$  and changes in real interest

rates  $A_t$  can be seen from Equation 7, slower growth favors a higher asset-to-income ratio, increased longevity favour a higher asset-to-income ratio, while the effect of a change in  $r$  is ambiguous as it depends  $\sigma$ .

## 2.5 Adding Pension benefits and clarifying valuation effects

In the previous section, we assumed that post retirement consumption depended only on assets accumulated during working years; we did not allow for pension benefits. In this section, we introduce pension benefits to show how this affects desired asset holding. We will focus on a constant interest rate setting throughout this section. What will be important here is to distinguish between total post-retirement wealth– inclusive of the value of one’s pension–, which we will denote by  $A_t$ , and accumulated wealth through savings decisions that we will continue to denote by  $a_t$ . So, at retirement, total wealth  $A_t$  will be equal to the accumulated wealth  $a_t$  plus the present value of one’s flow pension benefits  $pb$ . The accumulation equation for  $a$  post retirement becomes  $\dot{a} = a_t r + pb - c_t$  and the value function associated with retiring with  $a_t$  of accumulated asset and a flow pension benefit  $pb$  is then given by  $V(A_t) = \frac{(A_t)^{1-\sigma}}{1-\sigma} \Gamma(r) = \frac{(a_t + \frac{pb}{r+\delta_2})^{1-\sigma}}{1-\sigma} \Gamma(r)$ .

For a household in the active phase, the accumulation equation will be unchanged except that labour income  $w$  must now be interpreted as net of any deductions for pensions.<sup>42</sup> The Euler Equation associated with optimal consumption remains as before with  $A_t$  simply replacing  $a$ . The steady state expression for the asset-to-income ratio therefore remaining equal to

$$\frac{A}{y} = (\delta_1 q)^{\frac{1}{\sigma}} \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right]^{-1} [\rho + \delta_1 + \sigma g - r]^{\frac{-1}{\sigma}}$$

In brief, the introduction of pensions benefits does not affect the target level of assets-to-income once pension benefits are included into wealth calculations.

However, the presence of pension benefits does affect the steady state amount of wealth  $a$  accumulated through savings. Not surprisingly, expected higher pension payments decrease the need to accumulate wealth through savings. Assuming that a pension benefit is a fraction  $\alpha$  of labor market income  $w$ , the steady state value for the accumulated-asset-to-income ratio  $\frac{a}{y}$  is now given by by<sup>43</sup>

<sup>42</sup>In our simplified environment, note that we are interpreting the pension benefit as being independent of the length of time in employment. It is a payment that is conditional only on the health shock that placed someone into retirement.

<sup>43</sup>May want to add details, as done in previous note share on September 13



$$\frac{a}{y} = \frac{(r + \delta_2)\frac{A}{y} - \alpha}{r(1 - \alpha) + \delta_2}$$

If  $\alpha = 0$ , we are back to our previous expression for  $\frac{a}{y}$ . When  $\alpha > 0$ , the effect of a change in  $r$  on  $\frac{a}{y}$  can be decomposed in two parts. There is the effect of  $r$  on total desired total wealth-to-income  $\frac{A}{y}$  which reflects as before the competition between inter-temporal substitution motives retirement motives. However, there is now an additional effect due to the valuation of pension benefits. This later effect is always positive, that is, holding constant desired total wealth-to-income  $\frac{A}{y}$ , a lower interest rate environment reduces the incentive to accumulate wealth through savings. For example, when comparing two individuals with different pension coverage (different levels of  $\alpha$ ), this implies that a household with less pension coverage (lower  $\alpha$ ) will save more and be observed to accumulate more wealth. In such an environment, if  $r$  decreases and  $\sigma > 1$ , then low  $\alpha$  households may choose to save more while high  $\alpha$  households may save less. This mechanism may help explain some of the heterogeneity we document regarding how wealth-the-income changed differentially between income groups over the period of declining interest rates . Since our measure of wealth-to-income does not include social security wealth, and that social security replacement rates are decreasing in income, this mechanism suggests that a low interest rate environment could have lead higher income groups to increase there target wealth-to-income ratio while simultaneously causing lower income individuals to accumulate less observed wealth. <sup>44</sup> We will explore this possibility quantitatively in the next section.

### Valuations effects more generally

In our baseline model households had only one savings vehicle: a short term bond. In this section we will briefly discuss how the framework extends to allow for interest rate sensitive assets. Our goal is to illustrate why different valuations scenarios can have very different effects on consumptions decision depending on the source of the valuation . To this end, let us consider introducing a Lucas tree into our setting where the tree produce a flow  $f$  of goods every period. A household can now hold a combination of bonds and trees. If we denote by  $z_t$  the price of a unit of trees at time  $t$ , then arbitrage between the two assets will cause  $z_t$  to satisfy the following asset pricing relationship

$$\frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - (r_t + \omega),$$

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<sup>44</sup>Interestingly, this mechanism could explain how a lower interest rate environment could favour the growth in gross household sector debt, as low income household maintain mortgages financed by high income households.

and households will be indifferent between holding bonds or trees. The active household consumption Euler equation in this case can be re-written as

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

where  $\Omega$  denotes household wealth that can now be composed of bonds and trees. Accordingly, the desired steady state income-to-wealth ratio, where wealth  $\Omega$  includes both bonds and tree, maintains all previous properties.

Now we can use this framework to consider two different type of changes that can lead to valuation effects, that is, changes in *Omega* that is due to capital gains instead of savings. Consider an active household that is initially in steady state as presented in Figure XXX, with initial combined positive holding of both bonds and treed as given by  $\Omega_{0}$ . Now we want to contrast the household's response to a one time permanent change in  $f$  – an increase from  $f$  to  $f'$ – with a one time permanent change in  $r$ – a decrease from  $r$  to  $r'$ . In both cases, this will cause  $z$  to jump to its new equilibrium value of either  $\frac{f'}{r}$  or  $\frac{f}{r'}$ . So in both case, the impact effect is increased wealth due to a valuation effect. However, the impact of consumption can be very different even if the initial effect on  $\Omega$  is of the same size. The first case is presented in panel (a) of the figure. Since the change in  $f$  does not affect neither the  $\dot{c} = 0$  line nor the  $\dot{\Omega} = 0$  line, the valuation effect will lead to an upward jump in consumption as the household's asset to income ratio is above target, and the household can therefore takes advantage of the valuation effect to consume more and run down assets. However, in the second case, the effect of consumption with be more muted. In fact, it is not clear if the valuation effect will cause an increase or a decrease in consumption when it is driven by a change in  $r$ . As illustrated in panel (b), the change in interest rates now affects both the  $\dot{c} = 0$  line and the  $\dot{\Omega} = 0$  line, and if  $\sigma$  is sufficiently large, this can cause the new steady state level of desired wealth to be greater that the wealth effect generated by the fall in rates. Hence, in this framework, the consumption effect of a re-valuation of assets can be very different depending on the change giving rise to the increased valuation. In particular, valuations effects due to decreases in interest rates will have more muted and even possible perverse effects of consumption when compared to valuations effects that are due to asset providing a greater flow of income.

### 3 A quantitative exploration of the drivers of increased Wealth-to-Income

In this section we will use Equation 7 to explore different forces which could have lead to the observed within household increases in wealth-to-income ratio over the period 1989-

2019. We will mainly focus on the observed accumulation pattern of the 54-64 age group as they are the households most likely to have wealth levels close to the steady state level implied by Equation 7. We begin by abstracting from the heterogeneity across households, and focus on explaining the average outcomes.

We will examine the effect of three type of changes: changes in longevity, changes in growth rates and changes in interest rates. Our baseline set of parameters are set as follows, where a period is a year. The working life is set at 40 years so  $\delta_1 = .025$ ,  $\delta_{2,1989}$  is set at .058, which corresponds to the 17.25 life expectancy at 65 years in 1989. The fraction of people that make it retirement is set at  $q = .81$  to match overall life expectancy. The growth level  $g_{89}$  is set at .02. For the real interest rate in 1989, we consider two values, either .04 or .03. For 2019, we set the real interest rate  $r_{2019} = 0$ , reflecting the large fall in real interest rates between 1989 and 2019. To reflect increased longevity, we set  $\delta_{2,2019}$  to .051, which correspond to the life expectancy of 19.6 years in 2019. Finally we set the growth rate at .015 as this reflects the growth rate in per capita income in the US from 1989 to 2019.

Our aim will be to match the average (within group) levels of wealth to income in both 1989 and 2019. The theoretical level are given by

$$\frac{a_{89}^{a,ss}}{y} = (\delta_1 q)^{\frac{1}{\sigma}} \left[ \frac{\rho + \delta_{2,89}}{\sigma} + \frac{\sigma - 1}{\sigma} r_{89} \right]^{-1} [\rho + \delta_1 + \sigma g_{89} - r_{89}]^{\frac{-1}{\sigma}}$$

$$\frac{a_{19}^{a,ss}}{y} = (\delta_1 q)^{\frac{1}{\sigma}} \left[ \frac{\rho + \delta_{2,19}}{\sigma} + \frac{\sigma - 1}{\sigma} r_{19} \right]^{-1} [\rho + \delta_1 + \sigma g_{19} - r_{19}]^{\frac{-1}{\sigma}}$$

These levels of wealth-to-income will depend on the value of  $\sigma$  and  $\rho$  in addition to the parameters noted above. Since there is considerable uncertainty regarding the appropriate values for  $\sigma$  and  $\rho$ , our approach will be to search for  $\sigma \in [1, 3]$  and  $\rho \in [.02, .05]$  to match our empirical counterparts for  $\frac{a_{89}^{a,ss}}{y}$  and  $\frac{a_{19}^{a,ss}}{y}$  when all three sources of changes are allowed to happen at once. Then we look at the respective roles of changes in longevity, growth and interest rates (one by one) in explaining the observed changes holding  $\sigma$  and  $\rho$  at the values needed to explain the aggregate change. <sup>45</sup> Results from this exercise are presented in Table X.

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<sup>45</sup>Note that, from an ex-ante perspective, it is not clear if there exist  $\sigma \in [1, 3]$  and  $\rho \in [.02, .05]$  that we allow us to match the data.

Table 5  
Contribution of longevity, growth and interest rates to change in asset-to-income ratio  
1989-2019

8.3-11.3 target (36%)			
	longevity	growth	interest rates
% point contribution			
$\Delta r = .04$	4.8	8.3	22.1
$\Delta r = .03$	5.4	6.8	17.5
	$\sigma$	$\rho$	$\frac{a}{y_{89}} - \frac{a}{y_{19}}$
$\Delta r = .04$	2.19	.042	8.4-11.4
$\Delta r = .03$	2.19	.042	8.5-11.5

9.2-14.3 target (55%)			
	longevity	growth	interest rates
% point contribution			
$\Delta r = .04$	4.6	7.6	31.2
$\Delta r = .03$	4.5	5.5	35.9
	$\sigma$	$\rho$	$\frac{a}{y_{89}} - \frac{a}{y_{19}}$
$\Delta r = .04$	2.5	.034	9.35-14.34
$\Delta r = .03$	2.74	.049	9.55-14.35

In Table 5, we report results for 4 cases. These cases differ with respect to the size of the change in  $r$  considered, and with respect to the targeted values for  $\frac{a}{y_{89}}$  and  $\frac{a}{y_{19}}$ . In the top panel of the figure, we use the shift share analysis to establish the values of  $\frac{a}{y_{89}}$  and  $\frac{a}{y_{19}}$  we need to target. In the bottom panel we used the change in slope of the the wealth on income relationship to establish the targets. In each of these panels, we consider the case with the change real interest rates goes from 4% to 0, or 3% to 0. For each case, we report the parameters  $\sigma$  and  $\rho$  needed to approximately match the targets and the values of  $\frac{a}{y_{89}}$  and  $\frac{a}{y_{19}}$  implied by these parameters. For example, to match the within household change in wealth to income implied by the shift share analysis (8.3-11.3) when  $r$  falls by 4%, we need  $\sigma = 2.19$  and  $\rho = .042$ . Both these values are within the bounds typically associated with these parameters. When looking at the individual contributions of each of three sources to change – longevity, growth and interest rates– we see that longevity can explain a 4.8 % point increase in the wealth-to-income ratio, that is, it can explain a wealth-to-income increase from 8.3 to 8.7. The effect of reduced growth can explain a 8.3 % point increase. Both of these taken together explain less than half of what is implied by the shift-share analysis. The biggest explanatory power comes from the fall in interest rates, as this explains a 22.1 % point increase which is around 2/3 of the implied increase.

Overall, the patterns documented in In Table 5 suggest that the change in longevity

observed over 1989-2019 can explain around a 4 to 6% point increase in wealth-to-income ratio. Similarly, a potential fall in the per-capita growth rate of the economy from 2% and 1.5%, can explain another 5-8 % point increase in the ratio. Both of these forces are significant, but they fall short of offering an explanation to most of the increase in wealth-to-income observed for the 55-64 age cohort. Most of the explanatory power reflects the accumulation incentives induced by the fall in interest rates.

It is worth noting that the asset demand curves implied by values of  $\sigma$  and  $\rho$  reported in the table are C-shaped— since  $\sigma > 1$  – and that the "reversal rate"  $\bar{r}$  is generally above 4%, implying that the economy is essentially operating on the downward segment of the asset demand throughout this period.

### **3.1 Heterogenous effects of change in $r$ on different households: the potential role of social security**

Recall that in Section xx we documented that ,while average within group wealth to income ratio increased substantially over the period 1989 to 2019, many lower income households did not conform to this pattern. While this differential pattern can have different sources, in this sub-section our aim is to discuss the extent to social security may help explain the differential change. It is important to recognize that social security is a very redistributive program. For low income earners it offers a high replacement rate in retirement. In effect, the amount of income one receives from social security is almost constant across large segments of the population. For example, based on Health and Retirement study at the University of Michigan, XXX report that a retired high educated white workers in 2020 received on average 21 232 \$ while a retired low educated white worker received 19 627\$ a year. A similar small differential is reported between high educated and lower educated black workers. Based on this,

## **4 General equilibrium**

Our quantitative exploration suggests that the fall in interest rates from 1989 to 2019 likely played a prominent role in the observed within group increased in wealth-to-income ratios. The reason we put forward is the need to accumulate more wealth for retirement when interest rates are low . This is a consequence of a C-shaped demand for assets induced when inter-temporal and retirement motives compete. However, in all our discussion up to now, we have been treating interest rates as exogenous. In general equilibrium, interest rates will need to adjust to equate the supply and the demand for assets. The aim of

this section is to briefly highlight some of the potential general equilibrium implications of having households whose wealth targets may be C-shaped.

To look at the general equilibrium implications of C-shaped asset demands, we first need to verify whether the C-shape property we derived for active households in Section xx is likely to be maintained when we look at the aggregate demand for asset for the economy; where this aggregate demand needs to combine the demand emanating from both active and retired household. In Appendix xxx we show how we can use the model of Section xx to derive a steady state aggregate demand for assets. This aggregate demand, denoted  $A^d$ , is given by Equation 8.<sup>46</sup>

$$A^d = \frac{Yh(r)}{\phi} \left( \frac{\phi + (1 - \phi)g(r)}{\phi + (1 - \phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}} \right) \quad (8)$$

where  $Y$  is aggregate income,  $h(r) = (\delta_1 q)^{\frac{1}{\sigma}} \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma}}$ ,  $\phi = \frac{\delta_2}{\delta_1 q + \delta_2}$  and  $g(r) = \frac{\sigma \delta_2}{\rho + \delta_2 - r + \sigma \delta_2}$

It is worth noting that this representation for the aggregate asset demand should be viewed as mainly illustrative as its derivation involves assumptions on how inheritances are shared among new households and on who the government taxes. These assumptions are made to make aggregation tractable. Notwithstanding this caveat, Equation 8 is quite informative as it helps clarify why the steady state aggregate demand for asset likely echos the property derived for assets demand by active households. The term  $h(r)$  in Equation 8 is the identical term used to capture the steady state asset-to-income ratio of active households. Although retired households have different consumption-savings incentives than active households, the assets with which they start their retirement reflect the decisions of active household. This explains why the term  $h(r)$  plays such a prominent role in the aggregate demand for assets. In effect, the aggregate demand for assets tends to depart from that of active households only due to the term  $\frac{\phi + (1 - \phi)g(r)}{\phi + (1 - \phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}}$  in Equation ???. Although it is difficult to derive simple analytical properties for Equation 8, it is easy to explore its quantitative properties. In effect, when using the range of parameters presented in Section xx, we have found that the aggregated demand for asset given by Equation 8 mimics very closely the properties of  $h(r)$ , in particular it maintains a C-shape when  $\sigma > 1$ .<sup>47</sup> Accordingly, this motivates us to discuss general equilibrium implications under

<sup>46</sup>To express this aggregate demand in levels, we have assumed away aggregate growth. Introducing growth is straightforward but requires specifying level variables as variables de-trended by the growth path.

<sup>47</sup>Furthermore, when  $\sigma$  is in the 2-3 range, the inflexion point is around 3%.

the assumption that the aggregate demand for assets as given by Equation ?? is C-shaped.

To highlight some (steady state) general equilibrium implications, we now need to combine the aggregate demand for asset given by Equation 8 with an aggregate supply of assets. Since we aim here to highlight only quantitative implications of having a C-shaped asset demand, we can adopt a rather simple structure for the asset supply. To this end, let us allow for only two types of asset: a short term government bond in quantity  $B^{48}$  and a mass one of Lucas trees. Where the Lucas trees pay dividend  $f$ . By arbitrage, the price of the Lucas tree will be given by  $\frac{f}{r}$  in steady state and hence the aggregate supply of assets, denoted  $A^s$ , can be expressed as

$$A^s = B + \frac{f}{r} \quad (9)$$

This aggregate supply of asset can be thought as the bond equivalent supply of asset, where the Lucas trees are converted to bonds at price  $\frac{f}{r}$ . In this environment, aggregate income will be given by  $\frac{w}{\phi} + f$ , that is, labour income plus dividends from trees. The equilibrium determination of interest rates and of aggregate wealth ( $B + \frac{f}{r}$ ) is therefore found by equating Equations 8 and 9.<sup>49</sup> This equilibrium condition is illustrated in three Panels in Figure xxx, where we are assuming that  $\sigma > 1$  and accordingly the aggregate demand for wealth is depicted as being C-shaped. The aggregate supply for asset is a decreasing function interest rates as higher interest rates decrease the price of the Lucas trees. Panel (a) depicts the situation where the supply of assets is cutting the demand for asset in the upper portion of C-shaped demand curve. This can be thought as a conventional configuration where locally, the demand curve for asset is sloping up and the supply curve is sloping down as functions of interest rates. For this configuration, our framework has nothing especially new to highlight. The second panel depicts the case where the demand for assets is sufficiently strong, or the supply of safe asset equivalents are sufficiently weak, such that the aggregate supply of assets cuts the demand on the lower portion of the C-shape. This is the configuration we want to focus upon here. Finally, panel (c) depicts

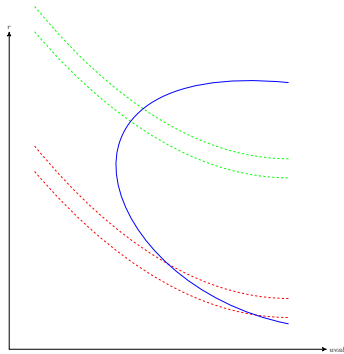
<sup>48</sup>In our derivation of the aggregate demand for assets, we imposed the assumption that any payments on government bonds was supported by a tax on active households

<sup>49</sup>The equilibrium condition for interest rates can be expressed as

$$\frac{B}{Y} + \frac{f}{Yr} = \frac{h(r)}{\phi} \left( \frac{\phi + (1 - \phi)g(r)}{\phi + (1 - \phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}} \right)$$

This formulation has the attractive feature of highlighting the potential role of the government debt-to-GDP ratio,  $\frac{B}{Y}$  and the capital share of income  $\frac{f}{Y} = \frac{f}{\frac{w}{\phi} + f}$  in potentially affecting interest rates.

Figure 7  
The perverse effects of changes in asset supply



the case where the supply curve cuts the demand curve more than once, which implies the existence of multiple steady state level of real interest rates. While this configuration opens the door to many intriguing issues, we will not pursue it further here.<sup>50</sup>

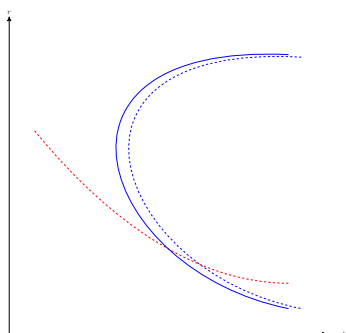
#### 4.1 Steady state general equilibrium comparative statics when asset demands are C-shaped.

Let us start by focusing on the configuration in panel (b) of Figure xx, and contrast how the effects of a reduction in asset supply – say due to a reduction in  $B$  – differs in this configuration relative to the case where the reduction happens when the economy is in the configuration of Panel (a). In both cases, a reduction in safe asset supply  $B$  will lead to a reduction in equilibrium interest rates. However, when in the configuration of panel (b), this reduction in supply actually induces a sufficient strong reduction in interest rates that, in equilibrium, total wealth increases. This increase in wealth arises despite the direct effect on wealth being negative. Such an outcome is potentially interesting given the large literature that pertains to explain the observed decrease in real interest rates in the first two decades of the 2000s as reflecting a decrease in the supply of safe assets (cite). If a reduction in the supply of assets occurs in the presence of an upward sloping demand for asset (upward as a function of  $r$ ), this would imply that total wealth should have been observed to decrease due to the reduction in safe assets. However, during the

<sup>50</sup>Some of the implications of multiple steady state interest rates are explored in – need to cite our previous paper BKM. In particular, this is discussed in the presence of sticky prices and implications for monetary policy are explored.



Figure 8  
The multiplier effects of changes in asset demand



period of decreasing interest rates in the 2000s, as we have discussed, total wealth increased substantially. A C-shaped asset supply offers a potential reconciliation whereby a reduction in safe asset supply could simultaneously explain the decrease in interest rates while at the same time being consistent with an induced valuation effect that is large enough to explain the observed increase in wealth.

The second general equilibrium implication of C-shaped demand curves we want to highlight relates to its potential role in amplifying demand changes on interest rates and wealth. For example, consider an increase in longevity, as captured by a shift in  $\delta_2$ . Increased longevity implies a rightward shift in the aggregate demand for assets. If this shift happens when the economy is operating in the configuration of panel (a) in Figure xxx, then the general equilibrium effect of such a change on household wealth holdings will be smaller than the partial equilibrium effect (ie when holding interest rates constant). However, as we noted in Section xxx, the partial equilibrium effect on asset demand of the increase in longevity observed over the last thirty years is likely rather small, so in general equilibrium, it would be even smaller. In contrast, if the economy is operating on the lower segment of a C-shaped asset demand, then even a small partial equilibrium on wealth demand can potentially cause a large change in interest rates and thereby induce a large change in individual level asset holdings. In this sense, while we found that from a partial equilibrium perspective increased longevity was unlikely the explanation for the observed within group increased wealth holdings, from a general equilibrium perspective, it may have played a larger role as it may be the source on the decrease in interest rates which we argued favored more accumulation of wealth.

In summary, from a general equilibrium perspective, C-shaped asset demand have the potential to better explain how decreases in safe asset supply and/or increases in factors such as longevity may have played a important role in causing large falls in interest rates while simultaneous explaining large increases in individual level wealth holdings. In both these cases, if the asset demand curve is simply the more conventional positive sloped function of interest rates, then such changes are unlikely to explain the join pattern of large falls in interest rates with substantial increases in within group holding of assets. <sup>51</sup>

## 5 Conclusion

A good comprehension of long run asset demands is necessary for a better understanding of low frequency movements in interest rates. In this paper we have used quantitative theory to support of the notion that long run asset demands are plausibly C-shaped and that the gradual fall in interest rates observed from the late 1980 up until the covid period likely favored an increased household demand for wealth. To support this point, we began by documenting how households – with the same demographics and income– appear to have targeted greater wealth-to-income ratios in 2019 than in 1989. We then presented a tractable model of wealth accumulation to explore different potential explanations to this observation. The model combines both inter-temporal substitution and retirement motives. An attractive feature of our simple setup is that it allows for straightforward quantification of competing mechanisms. While increased longevity and decreased aggregate growth can partly explain the observed increase in desired wealth levels, we found their effects to be rather small. In contrast, we found that the fall in real interest rates observed between 1989 and 2019 to be potentially of the right size to account for all of the observed increase wealth holdings. This quantitative finding arises when adopting estimates of the elasticity of inter-temporal substitution commonly found in the literature. Finally, on the aggregate front, we discussed how C-shaped asset demand can help explain, among others, how a decrease in the public supply of safe asset could have simultaneously caused decreases in interest rates and increases in overall wealth.

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<sup>51</sup>An alternative perspective on equilibrium determination in this economy is to look at it through the goods market. In this case, we can define a (steady state ) aggregate demand function  $AG^d$

$$AG^d = \left[ \frac{h(r)}{\phi} \left( \frac{\phi + (1 - \phi)g(r)}{\phi + (1 - \phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}} \right) \right]^{-1} \left[ B + \frac{f}{r} \right]$$

In the case where there is a unique equilibrium, as in panels (a) and (b) of Figure xxx, this function is monotonically decreasing in  $r$  even if asset demand are C-shaped. In the case of multiple equilibrium, it becomes non-monotonic.

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# Appendix

## A Data

For the main analysis we use four waves of the US Survey of Consumer Finances for 1989, 1992, 2016 and 2019. The 1989 and 2019 SCFs are used for the wealth-to-income ratio decomposition into between- and within-group components, while the 1989-1992 and 2016-2019 SCFs are used in the construction of saving rates corresponding to the beginning (1989) and the end (2019) of our period of interest for the joint analysis of changes in wealth-to-income ratios and saving rates. We further supplement the findings using SCF micro-data alone with the results that combine SCF with household-level aggregates reported in the US Flow of Funds Accounts and the National Income and Product Accounts.

Household wealth in the SCF is defined to include all assets of households (both real and financial) net of their liabilities. On the one hand, household non-financial (real) assets include primary and other residential real estate, non-residential real estate equity, as well as equity holdings in privately held businesses (both corporate and non-corporate) and other non-financial assets. Financial assets, on the other hand, include fixed-income assets, e.g. bonds, deposits, as well as mutual fund holdings, and directly and indirectly held stocks, and other financial assets. The split into fixed-income vs. equity components also covers defined contribution pensions of US households. While SCF collects information about the types of pensions households are entitled to (account or traditional pensions), the estimates of the wealth in defined benefit plans are not directly available. Given the importance of these plans in household pension wealth, we use estimates from [Sabelhaus and Volz \(2020\)](#) to construct a measure of wealth in SCF that includes defined benefit pensions, and use aggregate shares from detailed FFA pension accounts to split them into fixed-income vs. equity components, similar to defined contribution account pensions. Unlike other papers, we also do not exclude vehicles as a measure of consumer durables from household wealth in the SCF, given its importance for less wealthy households, which makes our measure of saving closer to the concept used by the Flow of Funds Accounts. On the liability side, we include both mortgage and non-mortgage household debt obligations.

When combining SCF with household-level aggregates from the Flow of Funds Accounts, we follow the literature in consistently defining detailed asset and liability classes in SCF and aggregate data, and then creating a larger number of asset/liability classes (see, for example, [Mian et al. \(2021b\)](#)), for which group ownership shares can be defined. The same grouping into a larger number of asset/liability classes is also useful for the construction of saving rates in raw SCF data, given that pure inflation factors from [Mian et al. \(2021b\)](#) are defined for the same asset and liability classes. We then construct each group's share in the total value of each asset/liability category and distribute FFA aggregates between groups using these shares. Each group's net worth is summed up using the values for each component. On the income side, we follow a similar approach by aggregating each group's income from its components, e.g., wages, business income, interest and dividend income, etc., which, in particular, allows us to be consistent with the balance sheet composition of households, at least on the asset side and the incomes generated by these assets. Similar to the assets/liabilities we do adjustments to the income components reported in SCF to make them consistent with their aggregate counterparts. See [Feiveson and Sabelhaus \(2019\)](#) for the discussion of the comparison between different components of wealth/income reported in FFA/NIPA and SCF.

When reporting results, we prefer using the SCF-based results given that they allow us to

construct consistent wealth-to-income ratios and saving rates (in particular, adjusting for net bequests, which can only be constructed in SCF) from the same data source. However, we also show that our wealth-to-income ratio decomposition results are largely unchanged when we use scaled SCF (aggregate) estimates, consistent with the literature. The scaled results in the aggregate do provide a better fit with the saving rates obtained from NIPA/FFA, which is why together with the main results for correlations between group-wise changes in wealth-to-income ratios and changes in saving rates using both raw and scaled SCF data, we provide additional evidence using scaled data as well.

Other data we use for the empirical analysis include pure price inflation factors from [Mian et al. \(2020\)](#), whose replication package provides them until 2016. We extend the series until 2019 using their methodology for different asset categories.<sup>52</sup> Since [Mian et al. \(2020\)](#) measures of wealth and saving do not include consumer durables, we also use an additional factor for consumer durables, and test the results for robustness to its different values.

## B Robustness results for the wealth-to-income ratio change decomposition

### B.1 Shift-share decomposition: alternative groupings

In this section, we present robustness results associated with using a different number of income-age groups (in [Table B1](#)) and using 2019 as a base-year (in [Table B2](#)) for the decomposition results.

Table B1  
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019: Robustness to number of age-income groups

Grouping	Total Change	Within, %	Between, %
10 inc gr × 6 age gr	2.82	59.4	40.6
12 inc gr × 6 age gr	2.82	54.9	45.1
15 inc gr × 5 age gr	2.82	51.8	48.2

The 10 income groups are defined as follows: 0-20, 20-40, 40-60, 60-80, 80-120, 120-160, 160-200, 200-250, 250-500, 500+ (000, in 2019 \$); while in the 12 income groups the top group is also split into the following additional groups: 500-750, 750-1250, 1250+ (000, in 2019 \$). The 15 income groups further split the top 1250+ bracket into 1250-1750, 1750-3000, 3000-15000, and 15000+ (000, in 2019 \$). The six age groups split the 65+ age category into 65-74 and 75+ years.

In the first panel of [Table B2](#) for comparison with [Auclert et al. \(2021\)](#) we present results for 12 age groups; in the second panel of the table we report results for different combinations of age and income groups.

<sup>52</sup>For the pure inflation factors on the liability side, however, we are unable to extend the series, and use the last available data point from 2016 for the additional years of interest.

Table B2  
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between  
1989 and 2019: Robustness to base year of income/wealth profiles

Definition	Total Change	Within	Between
		(%)	(%)
		12 Age groups	
1989 base	2.82	65.1	34.9
2019 base	2.82	52.1	47.9
		30 Income-age groups	
1989 base	2.82	61.6	38.4
2019 base	2.82	42.9	57.1
		60 Income-age groups	
1989 base	2.82	59.4	40.6
2019 base	2.82	45.5	54.5
		72 Income-age groups	
1989 base	2.82	54.9	45.1
2019 base	2.82	46.2	53.8
		75 Income-age groups	
1989 base	2.82	51.8	48.2
2019 base	2.82	42.8	57.2

## B.2 Regression-based decomposition approach

As the alternative approach to the simple shift-share decomposition presented in the main text, we use the 1989 cross section to estimate a wealth holding function, which we denote by  $F_{89}(age, y)$ , where as previously  $age$  represents the age of the household head and  $y$  represents real income of a household. Function  $F$  can take different forms. In this section, we focus on the polynomial function  $F$  in income and age.<sup>53</sup> Then, for each household in the 2019 cross section, we use estimated function  $F_{89}(age, y)$  to create a predicted wealth holding, which we denote by  $\hat{w}_{19}$ . These predicted wealth levels allow us to create a predicted wealth-to-income ratio in 2019 by adding up  $\hat{w}_{19}$  across households, and by dividing it by the aggregate income in 2019 (denoted  $\left(\frac{\hat{w}}{y}\right)_{19}$ ). By using the same prediction function for the wealth in 2019, as in 1989, the predicted ratio reflects only the changes in the proportions of different groups in the population. Accordingly, the fraction of the change in the wealth-to-income ratio explained by the within component can be expressed as

<sup>53</sup>We have run our predictive regressions using polynomials of order 3, 4, and 5. Polynomial function of order 5 delivers the best prediction. In Appendix ?? we show that these results are also similar to using a regression with a set of dummy variables for income and age groups, which we refer to as a step-function regression approach.



$$1 - \left[ \frac{\left(\frac{\hat{w}}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}}{\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}} \right]. \quad (\text{B1})$$

In Table B3, we also report the results of this exercise using our two measures of wealth, which both include defined benefit pensions, but differ in terms of the inclusion of the primary housing wealth. Using a fifth order polynomial in income and age to build predicted wealth, we find that the between component accounts for between 40 and 42 percent of the change in the aggregate wealth-to-income ratio, leaving the within component again accounting for slightly under 60 percent of the rise. While these findings still support an important role of changes in demographics and income inequality in explaining movements in the wealth-to-income ratio, they indicate that an even greater share is due to changes in wealth holdings keeping income and age constant.

Table B3  
Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Based on Regression

Definition	Total Change	Within	Between
		(%)	(%)
Wealth (baseline)	2.819	59.8	40.2
Wealth less housing	2.649	57.2	42.8

## C Proofs of Propositions and Lemmas

### C.1 Proof of Proposition 1

We first prove that asset holdings of active households converge to the long-run asset holdings  $a^{a,ss}(y, r)$  and then characterize the properties of  $a^{a,ss}(y, r)$ .

**Convergence of active households' asset holdings to  $a^{a,ss}(y, r)$ .** Let's recall the dynamics of the optimization problem

$$\dot{c}_t = \left( \frac{r_t - \rho - \delta_1}{\sigma} \right) c_t + \frac{c_t^{\sigma+1}}{\sigma} \delta_1 q a_t^{-\sigma} \Gamma_t^\sigma,$$

$$\dot{a}_t = r_t a_t + w_t - T_t - c_t,$$

Table B4

Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Using Step-function Regression Approach

Definition	Total Change	Within (%)	Between (%)
Wealth plus DB	2.819	64.8	35.2
Wealth plus DB less housing	2.649	63.7	36.3

Note: DB refers to the value of defined benefit pension schemes. The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the income groups (in thousands) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+.

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r_t \right].$$

Linearizing this system around the steady state ( $\dot{c}_t = 0$ ,  $\dot{a}_t = 0$ , and  $\dot{\Gamma}_t = 0$ ) with  $r_t = r$  leads to the dynamic system:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{a}}_t \\ \dot{\hat{\Gamma}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} \rho + \delta_1 - r & -\frac{c}{a}(\rho + \delta_1 - r) & \frac{c}{\Gamma}(\rho + \delta_1 - r) \\ -1 & r & 0 \\ 0 & 0 & \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{a}_t \\ \hat{\Gamma}_t \end{pmatrix},$$

where  $\hat{x}_t \equiv x_t - x$  means the deviation of a variable  $x_t$  from its steady state  $x$ , and  $\rho + \delta_1 - r = \delta_1 q c^\sigma a^{-\sigma} \Gamma^\sigma$ .

The determinant of the 3x3 Jacobian  $J$  is given by

$$\det(J) = (\rho + \delta_1 - r) \left( \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right) \left( r - \frac{c}{a} \right)$$

If  $r < \frac{c}{a}$ , then  $\det(J) < 0$ , implying that the steady state is saddle stable since  $\det(J) = \lambda_1 \lambda_2 \lambda_3$  and the eigenvalues ( $\lambda_1, \lambda_2, \lambda_3$ ) have opposite signs.

Combining  $\rho + \delta_1 - r = \delta_1 q c^\sigma a^{-\sigma} \Gamma^\sigma$  and  $\Gamma^{-1} = \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r$  leads to

$$\frac{c}{a} = \left[ \left( \frac{\rho + \delta_1 - r}{\delta_1 q} \right) \left( \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right)^\sigma \right]^{\frac{1}{\sigma}}.$$

Note that this equation also defines the implicit the long-run wealth-to-income holdings  $\frac{a^{a,ss}}{y}(r)$  where the disposable income  $y$  equals  $c$ .

Therefore, the convergence condition toward  $\frac{a^{a,ss}}{y}(r)$  is

$$r < \left[ \left( \frac{\rho + \delta_1 - r}{\delta_1 q} \right) \left( \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right)^\sigma \right]^{\frac{1}{\sigma}}.$$

This represents a necessary condition. A sufficient condition is

$$\max\{r, 0\} < \left[ \left( \frac{\rho + \delta_1 - r}{\delta_1 q} \right) \left( \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right)^\sigma \right]^{\frac{1}{\sigma}},$$

where  $\max\{r, 0\}$  guarantees consumption to be non-negative.

**Properties of  $\frac{a^{a,ss}}{y}(r)$ .** Recall the steady state wealth-to-income holdings

$$\frac{a^{a,ss}}{y}(r) = (\delta_1 q)^{\frac{1}{\sigma}} \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r \right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma}}.$$

Taking the derivative of  $\frac{a^{a,ss}}{y}(r)$  with respect to  $r$ , we have

$$\frac{d \frac{a^{a,ss}}{y}(r)}{dr} = (\delta_1 q)^{\frac{1}{\sigma}} (\rho + \delta_1 - r)^{\frac{-1}{\sigma} - 1} \left( \frac{1}{\rho + \delta_2 + (\sigma - 1)r} \right) \left[ 1 - \frac{\sigma(\sigma - 1)(\rho + \delta_1 - r)}{\rho + \delta_2 + (\sigma - 1)r} \right].$$

If  $\sigma \leq 1$ ,  $\frac{d \frac{a^{a,ss}}{y}(r)}{dr} \geq 0$  and hence the steady state asset holdings of active households are increasing in the interest rate.

Now let us assume that  $\sigma > 1$ . When  $r = \bar{r}$ , we have  $\frac{d \frac{a^{a,ss}}{y}(r)}{dr} = 0$  where

$$\bar{r} \equiv \frac{\sigma(\sigma - 1)(\rho + \delta_1) - (\rho + \delta_2)}{(\sigma - 1)(\sigma + 1)}.$$

If  $r > \bar{r}$ ,  $\frac{d \frac{a^{a,ss}}{y}(r)}{dr} > 0$ . And if  $r < \bar{r}$ ,  $\frac{d \frac{a^{a,ss}}{y}(r)}{dr} < 0$ . As a result,  $\frac{a^{a,ss}}{y}(r)$  is increasing (decreasing) in the interest rate when  $r$  is above (below)  $\bar{r}$ . Hence,  $\frac{a^{a,ss}}{y}(r)$  is C-shaped in the space  $(r, a)$ .

*Q.E.D.*

## D Appendix: Deriving the Steady State Aggregate demand for assets for both active and retired households

To derive the aggregate demand for asset, let us begin by normalize the population to have a measure 1 of households, with the implied fraction  $\phi \equiv \frac{\delta_2}{\delta_1 q + \delta_2}$  who are active and the fraction  $1 - \phi$  who are retired. When a household dies it is replaced by a new active household. Since we have not introduced annuity markets, private agents will have positive asset holdings when they die and therefore there will be unintended bequests. We assume that the unintended bequest of a household goes to the newborn household replacing that household. To keep the structure more tractable, we assume that the government ensures — through a tax  $T_{2t}$  on active households —

that all newborn households receive the same bequest.<sup>54</sup> Under this assumption, if asset holdings are equal across active households at a point in time, then the system inherits a representative agent structure for active households.<sup>55, 56</sup>

We can now determine total asset demands in this economy in a steady state with constant interest rates and taxes. This demand is comprised of both the long-run (per household) asset demand function of active households,  $a^{a,ss}(y, r)$ , and that of retired households, which in its aggregate will be denoted  $a^{r,ss}$ . For simplicity, let us consider the case with growth first. The steady state asset demand function of active households when interest rates are constant is given in Proposition 1 and can be stated as  $a^{a,ss}(y, r) = h(r)y$  where  $h(r) = \sigma(\delta_1 q)^{\frac{1}{\sigma}}(\rho + \delta_2 + (\sigma_2 - 1)r)^{-1}(\rho + \delta_1 - r)^{\frac{-1}{\sigma_2}}$ , with  $y = c$  in steady state. Since long-run asset demands relative to consumption of active households go to  $\infty$  when either  $r$  goes to  $\rho + \delta_1$  or  $-\frac{\rho + \delta_2}{\sigma - 1}$ , we restrict attention to situations where  $r \in \left(-\frac{\rho + \delta_2}{\sigma - 1}, \rho + \delta_1\right)$  as this is the only feasible range for a steady state equilibrium.

To get the steady state asset demand for retired households, we need to aggregate the asset holdings across the different retirement cohorts. With  $r < \rho + \delta_1 \leq \rho + \delta_2$ , retired households will be depleting their asset holdings as they age. In particular, this will cause the asset holdings of a retired household who retired  $\tau$  periods ago with  $a$  assets to be given by  $ae^{-\left(\frac{\rho + \delta_2 - r}{\sigma}\right)\tau}$ . Furthermore, note that the aggregate consumption of retirees satisfies the relationship  $c_t^r = a_t^r \Gamma^{-1}$ , where  $a_t^r$  is the total assets held by retirees at time  $t$ . Since in steady state, each retiree starts retirement with the same amount of assets, which is equal to the steady state asset holdings of active households ( $a^{a,ss}$ ), the aggregate asset holdings of retirees ( $a^{r,ss}$ ) is given by

$$a^{r,ss} = a^{a,ss}(y, r)(1 - \phi) \frac{\delta_2}{\frac{\rho + \delta_2 - r}{\sigma} + \delta_2} = a^{a,ss}(y, r)(1 - \phi)g(r)$$

where  $g(r) \equiv \frac{\delta_2}{\frac{\rho + \delta_2 - r}{\sigma} + \delta_2}$ .

As a result, total asset demand in the steady state of this economy can be expressed as

$$a^{t,ss}(y, r) = \phi a^{a,ss}(y, r) \left(1 + \frac{g(r)(1 - \phi)}{\phi}\right).$$

This expresses aggregate asset demand as a function of the total income of  $y$  of active households. However,  $y$  itself depends on  $a^s$ s and therefore to get an expression for the demand

<sup>54</sup>We are also assuming away pensions. This is without much loss since in general equilibrium, pension must be paid for and therefore are just an alternative form of accumulation.

<sup>55</sup>Assuming that active households act like a large family as in Gertler et al. (2020) would lead also to maintain the tractability of the representative agent structure.

<sup>56</sup>The tax on active household needed to ensure that that all newborns receive the same bequest is determined by the budget constraint

$$\delta_1(1 - q)\phi a_t + \delta_2(a^r) = [\delta_1(1 - q)\phi + \delta_2(1 - \phi)] a_t - \phi T_{2t}.$$

where  $a^r$  is aggregate holding by retirees. The first term on the left hand side of this equation is the total funds received from accidental bequests. On the right hand side, the first term is the funds needed to give to newborn active households while the second term is the tax levied on all active households to equalize wealth between newborns that inherited from retired and active households. Rearranging the equation, we obtain that  $T_{2t} = \delta_2(a_t^r)/\phi$ .

for households that depends on fundamentals, we need to use the goods market clearing condition. The goods market equilibrium condition is given by

$$\phi c = \phi w + f - a^r \Gamma^{-1}$$

where  $a^r \Gamma^{-1}$  is the aggregate consumption of retirees. Using good market equilibrium condition to replace  $c = y$  in the active household's asset demand, the aggregate asset demand can now be expressed as

$$a^T = h(r) \left( w + \frac{f}{\phi} \right) \left( \frac{\phi + (1 - \phi)g(r)}{\phi + (1 - \phi)g(r)(\delta_1 q)(\rho + \delta_1 - r)^{\frac{1}{\sigma}}} \right)$$

where  $\phi w + f$  is aggregate income. Recall that we have not introduced aggregate growth in this formulation. Allowing for aggregate growth is straightforward but require expressing aggregate asset demand as deviations from a growth path. <sup>57</sup>

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<sup>57</sup>The out of steady state aggregate dynamics for this economy can be represented by the as following set of equations. This formulation continues to assume the use of taxes that ensure that all new born households receive the same inheritance and that active households are taxed to pay for government spending and public debt payments.

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma} \delta_1 q V_a(a_t, \Gamma_t) \quad \dot{\Gamma}_t = -1 + \Gamma_t \left[ \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r_t \right] \quad \frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - r_t$$

$$\dot{a}_t = w + r_t a_t - T_t - c_t \quad \text{where taxes are} \quad T_t = \frac{(a_t - B - z_t)\delta_2}{\phi} + \frac{G + B r_t}{\phi}$$

plus the goods market clearing condition

$$\phi c_t = \phi w + f - G - (B + z_t - \phi a_t) \Gamma_t^{-1}$$

where  $c_t$  is the consumption of the representative active household and  $a_t$  is its asset holdings. The aggregate consumption of retirees is given by  $(B + z_t - \phi a_t) \Gamma_t^{-1}$  and its aggregate wealth holding is given by  $(B + z_t - \phi a)$