TESTING FOR UNIT ROOTS IN ECONOMIC TIME-SERIES WITH MISSING OBSERVATIONS

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ABSTRACT

This paper considers unit root testing of time-series data with missing observations. Three procedures for dealing with the gaps are discussed. These include: ignoring the gaps, replacing the gaps with the last available observation, and filling the gaps with a linear interpolation method. The tests for the first two procedures yield test statistics which have the same asymptotic distribution as that tabulated by Dickey and Fuller (1979) for the complete data situation. The remaining procedure yields a test statistic that has an asymptotic distribution that differs from Dickey and Fuller's tabulated distribution by an adjustment factor. In addition, models that include an ARIMA (0,1,q) error and augmented Dickey-Fuller tests are also considered in this paper. A simulation experiment is performed for the above models using the A-B sampling scheme. The results show that ignoring gaps in time- series data with missing observations produces unit root tests that are *more* powerful than the other two approaches that are considered.

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1. INTRODUCTION

This paper deals with the practical problem of testing for a unit root in time-series data when there are missing observations, or the data are irregularly observed. The situation of missing observations in time-series data is quite common, yet little attention has been given to it in the unit root literature. For example, series of trading data relating to financial assets (such as stocks or futures contracts) may have "gaps" in them for certain days. In some countries, for reasons relating to budget constraints or publication delays, it was the case historically that some economic time-series were published only annually (or quarterly), but more recently they have been published quarterly (or monthly). Of course, the underlying economic activity was always taking place at the higher frequency - it was simply not being recorded, and so the associated long-run measured series effectively has "missing observations", following a systematic pattern, in the earlier time periods. Finally, in many instances, we encounter economic time-series data which exhibit a small number of extreme outlier values of a type that clear reflect mis-recorded data, or once-and-for all events. Although a dummy-variables approach may be helpful in some such instances (*e.g.*, Perron (1989)), another approach is to remove such observations and treat them as "missing". These, and other related matters, are discussed in some detail in the context of forecasting stationary time-series by Harvey (1989, pp.325-341), for example. He also gives some interesting examples relating to electricity consumption data over a period including a miners' strike, and to time-series data for road deaths in the context of structural changes and outliers[1]. Harvey and Pierse (1984) also discuss the problem of missing time-series observations in the context of the Kalman filter, and illustrate their methods with the well-known Box-Jenkins (1976) airline passenger data.

The paper is divided into seven sections. Section 2 provides the background for some of the existing literature on testing for unit roots. An examination of previous related work on testing for a unit root under conditions of missing or irregularly observed data is covered in Section 3. Section 4 provides the relevant asymptotic theory associated with the problem being studied here. Section 5 elaborates on some extensions, and outlines the Monte Carlo simulations that we use to evaluate the finite sample characteristics of our procedures. The results of these simulations are discussed in Section 6, and the final section provides conclusions and some suggestions for further research.

2. TESTING FOR A UNIT ROOT

Many studies have considered different ways to test for the presence of a unit root. Dickey and Fuller (1979) derived the distribution for unit root test statistics when the *estimated* model is (a) with no drift and no trend; (b) with a drift and no linear trend; and (c) with both a drift and linear trend. The null disributions for the Dickey-Fuller test statistics are based on data generated by an autoregressive integrated moving average process of order (p,1,0). These tests must be reconsidered when the data contain a moving average component. In response to this limitation, many authors have explored extensions to Dickey and Fuller's work.

Dickey and Fuller (1981) tested various hypotheses of the existence of a unit root using the likelihood ratio test. The null hypothesis of a unit root is tested when the time-series Y_t is generated in the following manner:

$$Y_t = \alpha Y_{t-1} + u_t \quad , \tag{1}$$

with $\alpha=1$, $Y_0=0$, and $E(u_t)=0$, var $(u_t)=\sigma^2$. The three Dickey-Fuller (DF) regression equations are given as follows:

$$\begin{aligned} Y_t &= \alpha Y_{t-1} + u_t \\ Y_t &= \beta + \alpha Y_{t-1} + u_t' \\ Y_t &= \beta + \gamma t + \alpha Y_{t-1} + u_t'' \quad . \end{aligned} \tag{2}$$

equivalently, the fitted regressions can be written as

$$\begin{aligned} \Delta Y_t &= (\alpha - 1) Y_{t-1} + u_t \\ \Delta Y_t &= \beta + (\alpha - 1) Y_{t-1} + u_t' \\ \Delta Y_t &= \beta + \gamma t + (\alpha - 1) Y_{t-1} + u_t'' \quad , \end{aligned} \tag{3}$$

and then we test H_0:$(\alpha-1)=0$.

In this paper we are testing H_o: $\alpha=1$ in each case by using either the T(a-1) statistic or the $t_\alpha=(a-1)/se(a)$ statistic, where T is the sample size, a is some estimator of α, and se(a) is the standard error for a. There also exist corresponding joint tests of H_o: $(\beta,\alpha) = (0,1)$ in the drift/no-trend case and H_0:$(\beta,\gamma,\alpha) = (0,0,1)$ and H_0:$(\beta,\gamma,\alpha)=(\beta,0,1)$ in the drift/trend case, but these are considered here.

When time-series data are aggregated, often a moving average (MA) error term will arise. One way to deal with this is to include lagged values of the dependent variable in the DF regression. Said and Dickey (1984) derived a test for a unit root based on an approximation of an autoregressive-moving average model by an autoregression. The autoregressive model suggested by Said and Dickey, in the general drift/trend case, is as follows:

$$\Delta Y_t = \beta_0 + \gamma t + (\alpha-1)Y_{t-1} + \sum_{i=1}^{k} \beta_i \Delta Y_{t-i} + \epsilon_t \quad , \qquad (4)$$

where ϵ_t is iid $(0, \sigma^2)$.

Dickey and Said (1981) discussed testing for a unit root with the hypothesis H_o: $\alpha-1=0$ when the orders of the moving average and autoregressive process (p,q) are known. However, p and q are typically *unknown* and one must determine α prior to estimating p and q. Said and Dickey used least squares to estimate the coefficients in the autoregressive model and found that if k approaches infinity as T approaches infinity, the resulting statistics have limiting distributions identical to those given by Fuller (1976). Therefore, it is possible to determine the existence of a unit root without knowing p and q, and some of the "pre-test" implications of this are explored by Dods and Giles (1995), and others.

Phillips (1987) showed that least squares regression consistently estimates a unit root in spite of the presence of autocorrelated errors or heteroscedasticity, and he also derived the limiting distributions of both the ordinary least squares (OLS) estimator of α and its "t-statistic" in the no-drift/no-trend case in (2), under very special error structures. The normalized limiting distribution of the OLS estimator of α, say $\hat{\alpha}$, is:

$$T(\hat{\alpha}-1) \Rightarrow \frac{\frac{1}{2}(W(1)^2-(\sigma_u)^2/\sigma^2)}{\int_0^1 W(r)^2 dr}$$

where $W(r)$ *is a standard Wiener process, so* $W(1)$ *is* $N(0,1)$, **(5)**

$$(\sigma_u)^2 = \lim_{T\to\infty} T^{-1} \sum_1^T E\,(u_t^2)\ ,$$

$$\sigma^2 = \lim_{T\to\infty} E\,(T^{-1}\sum_1^T u_t)^2\ .$$

The symbol "⇒" is used to signify the weak convergence of the associated probability measures as $T\uparrow\infty$. The quantities concerned ($T(\hat{\alpha}-1)$, for example) converge to a random function, and the integrals relate to functionals of Wiener processes.

When the innovation of u_t is iid$(0,\sigma^2)$, then $\sigma_u^2=\sigma^2$, which leads to the following representation:

$$T(\hat{\alpha}-1) \Rightarrow \frac{\frac{1}{2}(W(1)^2-1)}{\int_0^1 W(r)^2 dr}\ . \qquad (6)$$

Phillips (1987) showed that the limiting distribution of the (suitably standardized) OLS estimator of α has the same general form for a wide class of innovation processes, $\{u_t\}$. Similarly, the limiting distribution of t_α for testing H_o: $\alpha=1$ also depends on the variance ratio σ_u^2/σ^2:

$$t_\alpha \Rightarrow \frac{\frac{\sigma}{2\sigma_u}(W(1)^2-\sigma_u^2/\sigma^2)}{\{\int_0^1 W(r)^2 dr\}^{1/2}}\ . \qquad (7)$$

In the iid case, $\sigma=\sigma_u$ and (10) simplifies accordingly.

Hamilton (1994) presents the limiting distributions of the normalized OLS estimator and t-statistic for the drift/no-trend case:

$$
\begin{aligned}
T(\hat{\alpha}-1) &\Rightarrow \frac{\frac{1}{2}\{[W(1)]^2-1\}-W(1)\cdot\int_0^1 W(r)dr}{\int_0^1 [W(r)]^2 dr-[\int_0^1 W(r)dr]^2} \\
t_{\alpha} &\Rightarrow \frac{\frac{1}{2}\{[W(1)]^2-1\}-W(1)\cdot\int_0^1 W(r)dr}{\{\int_0^1 [W(r)]^2 dr-[\int_0^1 W(r)dr]^2\}^{1/2}}
\end{aligned}
\qquad (8)
$$

and the drift/trend case:

$$
\begin{aligned}
T\ (\hat{\alpha}-1) &\Rightarrow \frac{\frac{1}{2}([W(1)]^2-1)-W(1)A}{D} \\
t_{\alpha} &\Rightarrow \frac{\frac{1}{2}([W(1)]^2-1)-W(1)A}{D^{1/2}} \quad ,
\end{aligned}
\qquad (9)
$$

where, $A=[4\int W(r)dr-6\int rW(r)dr]-[W(1)-\int W(r)dr][12\int rW(r)dr-6\int W(r)dr]$, $D=[\int[W(r)]^2dr+12\int W(r)dr\int rW(r)dr-4[\int W(r)dr]^2-12[\int rW(r)dr]^2$, all integrals are over the range [0,1], and to simplify the expression, we have assumed iid errors. For finite samples, the MacKinnon (1991) percentiles, based on Monte Carlo simulations, may be used.

Phillips and Perron (1988) extended the results of Phillips (1987) to cases where (i) a drift and (ii) a drift and a linear trend, with the presence of autocorrelated errors, are included in the analysis. In Phillips (1987), test statistics and limiting distributions were derived that included a transformation using the Newey-West (1987) consistent estimator of the error covariance matrix. These test statistics were also used in the Phillips-Perron extended analysis.

The results obtained indicated that the Phillips-Perron appraoch can be advantageous for positive (as signed in equation (53) below) moving average errors. The powers associated with this test were better than the ones examined by Dickey and Said (1981). However, with negative moving average errors, the Said-Dickey procedure was recommended. Since negative MA errors are likely to arise in practice (Dods and Giles (1995) and Schwert (1989)), this result has important practical implications.

Hall (1989) tested for unit roots in a time-series which was generated with an MA component of order q, using an instrumental variable (IV) approach. In particular, he used Y_{t-k} (for k>q) as the instrument for Y_{t-1} and considered the three regression equations analyzed by Phillips and Perron (1988). This form of estimation allowed Hall to use Dickey and Fuller's (1979) tabulations directly because the asymptotic bias caused by the correlation between Y_{t-1} and the error term, u_t, was not present. Hall's results indicated that the IV tests exhibited less size distortion in finite samples than the Phillips-Perron tests. For negative (in the sense of equation (53) below) MA errors, this difference could be substantial. Pantula and Hall (1991) extended Hall's work by considering data generated by an ARIMA (p,1,q) model:

$$\begin{aligned} Y_t &= Y_{t-1} + Z_t \\ Z_t &= \sum_{j=1}^{p} \theta_j Z_{t-j} + u_t \\ u_t &= \sum_{i=1}^{q} \psi_i \epsilon_{t-i} + \epsilon_t \quad . \end{aligned} \tag{10}$$

To estimate the model in equation (10), they considered:

$$Y_t = \alpha Y_{t-1} + \sum_{j=1}^{p} \theta_j Z_{t-j} + u_t \quad . \tag{11}$$

As u_t is a moving average process of order q, the regressor variables Y_{t-1} and Z_{t-j} are correlated with u_t , so (for a *fixed* lag length, p) the OLS estimators of α and θ will be inconsistent and their distributions will not converge to the usual Dickey-Fuller distribution.

Pantula and Hall found that IV estimators performed well in relation to Said and Dickey's results. They also found that overspecifying p and/or q does not have an effect on the power of the unit root tests, but underspecifying p and/or q leads to inconsistent estimators and unacceptable test properties.

Hall (1992) tested for unit roots using instrumental variable estimation when (p,q) were chosen from the data in a "pre-test" manner. He considered the use of the Pantula-Hall test statistics after (p,q) had been estimated from the data. To estimate (p,q), Hall derived a set of residuals from the following three regressions: (a) random walk; (b) drift but no linear trend; and (c) drift with a linear trend. He also provided conditions under which test statistics based on the estimates of (p,q) converge to the Dickey-Fuller distribution. Hall concluded that if the estimates of (p,q) are asymptotically independent of the unit root test statistic and the limiting probability of under-selecting either p or q is zero, then the Pantula-Hall test statistic converges to the appropriate Dickey-Fuller distribution. Further, when (p,q) are estimated from the data the unit root test performs well for moderate to large sized T.

Hall (1994) tested for a unit root in an autoregressive integrated time-series $(p_0,1,0)$ with pretest data-based selection of p_0, and he examined the impact of this selection ADF test. He found evidence that there may be power gains from choosing p from the data, and he derived conditions under which the ADF statistic converges to the appropriate Dickey-Fuller distribution. These conditions hold true when p is chosen using a general to specific strategy, the Hannan and Quinn (1979) information criterion, or the Akaike (1973) information criterion. More specifically, this unit root test should be calculated with $p \geq p_o$, and then the limiting distribution of the test statistic, given an estimated value of p=j, is that of the ADF statistic when p is set *a priori* to $j \geq p_o$. Otherwise the ADF statistic will not converge to the usual Dickey-Fuller distribution.

3. MISSING OR IRREGULARLY OBSERVED DATA

Although the above studies deal with testing for a unit root under different circumstances, they do not address the problem of time-series with missing observations. Often a time-series does not contain observations at consecutive time periods. For instance, data gathered from the stock market is a prime example of a series with missing observations. With the stock market being closed on holidays and weekends, gaps are generated in stock market data. Another practical illustration, and one that will be presented here, is when the frequency of data collection is altered. Suppose the frequency is changed from quarterly to monthly, in the case of a stock (rather than flow variable). The resulting series will have two missing monthly observations each quarter in the earlier part of the series.

The general topic of missing observations in the analysis of stationary time-series data has been examined by, for example, Little and Rubin (1983, 1987), Harvey and Pierse (1984), Kohn and Ansley (1984), and Harvey (1989). However, testing for a unit root in a time-series with missing observations has received little attention: it has been addressed tangentially in earlier work by Savin and White (1978) and Bhargava (1989). Savin and White (1978) considered three procedures for testing for a first-order autoregressive process in regression errors when there are missing observations. The procedures included the Durbin-Watson test, tests based on a set of uncorrelated residuals, and large sample likelihood ratio and Wald tests. Savin and White found that ignoring the missing observations left the null distribution unaltered when computing the usual Durbin-Watson d statistic.

Savin and White considered a model with m consecutive missing observations:

$$Y^{*}=\begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix}=\begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}\boldsymbol{\beta}+\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}=X^{*}\boldsymbol{\beta}+u^{*} \quad , \qquad (12)$$

where $y^{(1)}$ is an $(n_1 \times 1)$ $(n_1 > 0)$ vector of consecutive observations of the dependent variable before the gap, $y^{(2)}$ is an $(n_2 \times 1)$ $(n_2 > 0)$ vector of consecutive observations of the dependent variable after the gap, X^* is an $(n \times k)$ matrix of $n(= n_1 + n_2)$ observations on k fixed regressors, partitioned conformably with Y^*, and u^* is an $(n\times 1)$ vector of disturbances partitioned conformably with Y^*. The null hypothesis of H_0: $\rho=0$ was tested against H_1: $\rho>0$ when the elements of u^* were generated by the first order autoregressive process:

$$u_t^* = \rho u_{t-1}^* + \epsilon_t \quad . \tag{13}$$

A Durbin-Watson type statistic was calculated of the following form:

$$d' = \frac{z_*' A z_*}{z_*' z_*} \quad , \tag{14}$$

where z_* was the vector of OLS residuals computed from a sample with missing observations and A is the usual differencing matrix. In addition, a second Durbin-Watson statistic, d^*, was defined as:

$$d^* = \frac{z_*' A_* z_*}{z_*' z_*} \quad , \tag{15}$$

where the numerator omits the square of the difference across the gap. Therefore, it follows that $d^* \leq d'$. It was found that for certain values of ρ and m the statistic d^* appeared preferable to d'.

The main advantage of using d' as opposed to d^* is that it has the same null distribution as the usual Durbin-Watson statistic in the absence of a gap in the data. Hence, existing tables can be used for a bounds test of H_o when there are missing observations. The same point applies if the exact Durbin-Watson test is used, rather than the bounds test, with critical values computed for a given X matrix using, for example the algorithm of Davies (1980) as in the DISTRIB command in the SHAZAM (1997) package.

Bhargava (1989) tested for serial independence with missing observations with the Durbin-Watson test. He also considered the classical regression model:

$$y = X^{*}\beta^{*} + u \quad , \qquad u = \rho u_{-1} + \epsilon \quad , \tag{16}$$

where y is a $(T\times1)$ vector of observations on the dependent variable; X^{*} is a $(T\times(k+1))$ matrix of independent variables, the first being a constant term; β^{*} is the parameter vector to be estimated; the elements of ϵ are white noise error terms with variance σ^2; so each element of u is distributed with zero mean and variance $\sigma^2/(1-\rho^2)$. It is assumed that m observations are missing and that there are q 'gaps' of length m_i (i = 1,...q) in the data, with T_i (i = 1,...q+1) observations available in each of the subsamples, so that $T = T_1+T_2+...T_{q+1}$.

Bhargava allowed for two different Durbin-Watson statistics, d' and d^{*}, depending on whether the gaps in the series were ignored. As noted by Savin and White, the bounds for the test statistic d^{*} will be different from those tabulated for d'. These bounds for d^{*} depend on the position of the gaps in the data and must be calculated separately in different applications. However, a bounds test based on the d' statistic is very convenient, so it is important to examine the consequences of ignoring the gaps in the data for the power function of the d' test.

The study found that if the power functions of any of these tests are discontinuous at any value of ρ in [0,1], then it is desirable to avoid the use of that particular test. However, provided that a constant term is included in the regression model, the power functions of both d' and d^{*} are continuous in [0,1] even in the presence of gaps in the data. So, the use of the usual Durbin-Watson test in the presence of occasional gaps in the data is reasonable.

There has been only a very limited discussion of the effects of missing data on unit root tests. Shin and Sarkar (1993) tested for a unit root in an AR(1) time-series using irregularly observed data. They considered the OLS estimator, the one-step Newton-Raphson estimator and an OLS- type estimator which is a simple approximation of the Newton-Raphson estimator. When a unit root was present ($\rho = 1$) in $Y_t = \rho Y_{t-1} + \epsilon_t$, Shin and Sarkar found that the limiting distributions and corresponding t-statistics when ignoring the gaps were identical to those based on a complete series.

Hence, the limiting distributions and t-statistics did not depend on the sampling pattern, and asymptotic testing for a unit root with irregular samples can be carried out using the tables given by Fuller (1976). This result also applies to the more general autoregressive models:

$$Y_t = \mu + \rho Y_{t-1} + \epsilon_t^{\prime} \\ Y_t = \mu + \beta t + \rho Y_{t-1} + \epsilon_t^{\prime\prime} \quad . \tag{17}$$

Shin and Sarkar (1993) also considered replacing each gap in the irregularly observed time-series with the latest available observation, and they investigated the finite-sample properties for both ways of dealing with the missing observations in the case of the "A-B sampling scheme". A is the number of available observations and B is the number of missing observations, and the values of A and B are repeated for the length, m, of the time-series. Ten thousand replications were simulated for different values of m, A, B, and ρ and the tests were applied at the 5% significance level. This study, and those noted below, focussed primarily on size-distortion. The latter was *not* taken into account in the very limited power calculations that were undertaken.

The tests for the models outlined in (17) had smaller powers compared to the simple AR(1) model without (μ, β). This result was expected because the computation of the powers and the t-statistics involved the estimation of μ and (μ, β) respectively. Finally, in most cases it was found that the effect of missing observations on the powers of the tests was minor if the sample size was around 250 observations.

Shin and Sarkar (1994a) considered unit root tests for the ARIMA (0,1,q) model

$$Y_t = \rho Y_{t-1} + \epsilon_t + \beta_1 \epsilon_{t-1} + \ldots + \beta_q \epsilon_{t-q} \quad , \tag{18}$$

with irregularly observed samples. The unit root test proposed in this study was based on the instrumental variable estimation technique developed by Hall (1989, 1992). Again these tests were applied in the context of the usual three DF regressions - no-drift/no-trend, drift/no-trend, and drift/trend.

Instrumental variable estimators for ρ (the unit root coefficient) were defined for all three DF regressions, with q set to one and β_1, in (18), varied between positive and negative values. The empirical sizes[2] for the tests when A-B = 4-3, 5-2, and 6-1 were comparable to those when there were no missing observations. For negative values of β_1, the empirical sizes for the tests in the missing data cases were even better than those in the non-missing case. However, when $\beta_1 = -0.8$ all of the sizes were very different from the nominal level of 5%. Again, it was found that the model without a drift and a trend outperformed the other two models.

Finally, Shin and Sarkar (1994b) evaluated a likelihood ratio type unit root test for AR(1) models with nonconsecutive observations. They considered AR(1) models that contained: (i) a drift/no-trend and (ii) a drift/trend. The A-B sampling scheme was again utilized for different sample sizes. The sizes of the tests for A-B = 6-1, 5-2 were similar to those for the case where there are no missing observations, *i.e.*, A-B = 7-0. However, for A-B = 4-3 the empirical sizes were somewhat different, as compared with the A-B = 7-0 case. Shin and Sarkar concluded that the effect of missing observations on the sizes of the tests appeared to be small.

In this paper we extend Shin and Sarkar's studies in several directions. First. we consider another way of filling the gaps in the series: linearly interpolating between two observed values. As well as to cosidering empirical sizes, we also determine the size-adjusted powers associated with the modified DF tests. Finally, we consider the augmentation of the DF regressions to allow for MA errors in the irregularly observed data.

4. ASYMPTOTIC RESULTS FOR UNIT ROOT TESTS WITH MISSING OBSERVATIONS

This section defines the asymptotic distributions and "t-statistics" for the three usual DF models for four different cases: (i) series with no missing observations; (ii) closed series (*i.e.*, gaps ignored); (iii) gaps filled with last available observation; (iv) gaps filled by linear interpolation.

Case (iv) has not been considered previously. First, we assume that the DF regressions are fitted without "augmentation", but we relax this assumption subsequently

4.1 No-Drift/No-Trend

Following the Shin-Sarkar notation, the Data-Generating Process (DGP) and the fitted model are an AR(1) model $y_t = \rho y_{t-1} + u_t$ with $\rho = 1$, $y_o = 0$, and $E(u_t) = 0$, for all t. The unit root coefficient, ρ, corresponds to α in our earlier discussion. The length of the gap in a series is defined as $\Delta(k) = t_k - t_{k-1}$, where t denotes the passage of time in the generation of the series and t_k denotes the timing of the k'th observation available on the series with the incomplete data.

(i) <u>Series with No Missing Observations.</u>

The asymptotic distribution of the normalized OLS estimator of ρ in the case of a *complete* series (Phillips (1987)) is :

$$T(\hat{\rho}-1) \Rightarrow \frac{\frac{1}{2}[W(1)^2 - \sigma_u^2/\sigma^2]}{\int_0^1 [W(r)]^2 dr} \quad , \tag{19}$$

$$where \quad \sigma^2 = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E(u_t^2)$$

$$\sigma_u^2 = \lim_{T\to\infty} E\left[\frac{1}{T}\left(\sum_{t=1}^{T} u_t\right)^2\right] \quad ,$$

and $\hat{\rho}$ is the OLS estimator of ρ. If the u_t's are iid$(0,\sigma^2)$, this simplifies to:

$$T(\hat{\rho}-1) \Rightarrow \frac{\frac{1}{2}[W(1)^2 - 1]}{\int_0^1 [W(r)]^2 dr} \quad , \tag{20}$$

which is analogous to the limiting distribution in equation (5).

The limiting distribution for the t-statistic in this case is just that derived by Phillips (1987):

$$t_\rho \Rightarrow \frac{\frac{1}{2}\left(\frac{\sigma}{\sigma_u}\right)[W(1)^2 - \sigma_u^2/\sigma^2]}{\left[\int_0^1 W(r)^2 dr\right]^{\frac{1}{2}}} \quad . \tag{21}$$

If the errors are iid, then $\sigma_u=\sigma$ and (21) simplifies to:

$$t_\rho \Rightarrow \frac{\frac{1}{2}[W(1)^2 - 1]}{\left[\int_0^1 W(r)^2 dr\right]^{\frac{1}{2}}} \quad . \tag{22}$$

(ii) <u>Closed Series</u>

In this case, suppose there are n recorded observations for Y_t. From Shin and Sarkar (1993), using results derived by Phillips (1987), the asymptotic distribution of the normalized OLS estimator of ρ is :

$$\begin{aligned} n(\bar{\rho}-1) &\Rightarrow \frac{\frac{1}{2}[W(1)^2 - \sigma_1^2/\sigma_2^2]}{\int_0^1 [W(r)]^2 dr} \\ \text{where } \sigma_1^2 &= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=2}^{n} E\ (u_k^2)\right] \\ \sigma_2^2 &= \lim_{n\to\infty}\left[E\left(\frac{1}{n}\sum_{k=2}^{n} u_k\right)^2\right] \quad , \end{aligned} \tag{23}$$

where $n < T$ because the series is closed, $\bar{\rho}$ is the OLS estimator of ρ in the closed series, and $u_k = y_{tk} - y_{tk-1}$. Again, if the u_t's are independent then $\sigma_1^2/\sigma_2^2 = 1$ and the asymptotic distribution is unchanged from (20) for case (i) above. In fact, $\sigma_1^2=\sigma^2$ and $\sigma_2^2=\sigma_u^2$.

The limit distribution of the t-statistic, under independence, is that in (22):

$$t_\rho \Rightarrow \frac{\frac{1}{2}[W(1)^2-1]}{[\int_0^1 W(r)^2 dr]^{1/2}} \quad . \tag{24}$$

(iii) <u>Gaps Filled with Last Available Information</u>

Let $x_t = y_{tk-1}$, for $t_{k-1} \leq t < t_k$, $k = 2,\ldots,n$ and $x_{tn} = y_{tn}$. Further, let $a_t = x_t - x_{t-1}$ for $t = t_1+1, t_1+2,\ldots,t_n$. The new series, $\{x_t, t = t_1,t_1+1,\ldots,t_n\}$ can be treated as a complete data set where the missing values, for $t \neq t_1,t_2,\ldots,t_n$, are replaced with the latest available observations. Therefore, when $\rho = 1$, $x_t = x_{t-1} + a_t$, and there are now again T "observed" data points. The two methods of dealing with the gaps yield varying data sets which results in different variances for the error term.

The asymptotic results for the complete data series, x_t can be taken directly from Shin and Sarkar (1993). The limit distribution for the normalized OLS estimator of ρ is :

$$T(\ddot{\rho}-1) \Rightarrow \frac{\frac{1}{2}[W(1)^2-\sigma_3^2/\sigma_4^2]}{\int_0^1 [W(r)]^2 dr}$$

$$\textit{where } \sigma_3^2 = \lim_{T\to\infty}[\frac{1}{T}\sum_{t=2}^{T} E(a_t^2)] \tag{25}$$

$$\sigma_4^2 = \lim_{T\to\infty}[E\ [\frac{1}{T}(\sum_{t=2}^{T} a_t)^2]] \quad ,$$

where $\ddot{\rho}$ is the OLS estimator of ρ when the gaps are filled with the last available observation. If a_t is iid $(0,\sigma^2)$, then $\sigma_3^2 = \sigma_4^2 = \sigma^2$ and we get the usual asymptotic distribution, as in (23). Similarly,

$$t_\rho \Rightarrow \frac{\frac{1}{2}\left(\frac{\sigma_4}{\sigma_3}\right)[W(1)^2 - \sigma_3^2/\sigma_4^2]}{\left[\int_0^1 W(r)^2 dr\right]^{\frac{1}{2}}} \quad . \tag{26}$$

Under independence,

$$t_\rho \Rightarrow \frac{\frac{1}{2}[W(1)^2 - 1]}{\left[\int_0^1 W(r)^2 dr\right]^{\frac{1}{2}}} \quad , \tag{27}$$

which is just the usual result, (22).

(iv) <u>Gaps Filled with Linear Interpolation</u>

This method of filling the gaps has not been considered previously, but we have considered it here because it has intuitive appeal, and it seems to be somewhat more "refined" than approaches (ii) or (iii) above. For instance, if a series has two observations with three gaps in between, it could be argued that instead of using the last available observation to fill these gaps, a linear interpolation between the known observations could provide a "smoother" way of dealing with the gaps. If the previous known observation is far less (or greater) than the next, filling the gaps with this value will result in a series with a large "jump". Therefore, it seems sensible to use a technique that "smooths" the series from the first known low (high) observation to the next high (low) value. However, the distributional implications of such a procedure require careful consideration, even in large samples.

Let $x_{tk}' = y_{tk}$ for k=2,...,n. Also let $x_t' = x_{t-1}' + (\Delta(k))^{-1}(y_{tk} - y_{tk-1})$ for $t_{k-1} < t < t_k$; k = 2,...,n. Defining $a_t' = (\Delta(k))^{-1}(y_{tk} - y_{tk-1})$ yields the following relationship:

$$x_t' = x_{t-1}' + a_t' \quad . \tag{28}$$

Now, the asymptotic distribution of $T(\tilde{\rho} - 1)$ in this case is :

$$T(\tilde{\rho}-1) \Rightarrow \frac{\frac{1}{2}[W(1)^2 - \sigma_5^2/\sigma_6^2]}{\int_0^1 [W(r)^2 dr]} \tag{29}$$

$$\text{where } \sigma_5^2 = \lim_{T\to\infty}[\frac{1}{T}\sum_{t=2}^{T} E(a_t'^2)]$$

$$\sigma_6^2 = \lim_{T\to\infty}[E\ [\frac{1}{T}(\sum_{t=2}^{T} a_t')^2]] \quad ,$$

and $\tilde{\rho}$ is the OLS estimator of ρ when the gaps in the series are filled via linear interpolation, $a_t' = x_t' - x_{t-1}'$ and $\{x_t'\}$ is the linearly interpolated series. If we compare this situation with that in case (iii) above, we see that there $a_t = 0$ for all t associated with unobserved data points. So, in the iid case, $\sigma_3^2 = \sigma_4^2$, as noted previously. In contrast, here it is the case that $a_t' \neq 0$ for all such t. However, $\sum a_t = \sum a_t'$, and so $\sigma_6^2 = \sigma_4^2$. On the other hand, $\sum(a_t'^2) = (\Delta(k)^{-1})\sum(a_t)^2$, which implies that $\sigma_5^2 = \sigma_3^2(\Delta(k))^{-1}$. So, in general, (29) may be written as:

$$T(\tilde{\rho}-1) \Rightarrow \frac{\frac{1}{2}[W(1)^2 - (\frac{\sigma_3^2}{\sigma_4^2})\Delta(k)^{-1}]}{\int_0^1 [W(r)]^2 dr} \quad , \tag{30}$$

and this may be expressed in the form:

$$T(\tilde{\rho}-1) \Rightarrow AD(T(\hat{\rho}-1)) + \frac{\frac{1}{2}(\frac{\sigma_3^2}{\sigma_4^2})(1 - \frac{1}{(\Delta(k))})}{\int_0^1 [W(r)]^2 dr} \quad , \tag{31}$$

where $AD(T(\hat{\rho}-1))$ is the asymptotic distribution derived in (20). Now, if a_t' is iid $(0,\sigma^2)$, then $\sigma_3^2 = \sigma_4^2$, and (31) becomes:

$$T(\tilde{\rho}-1) \Rightarrow AD(T(\hat{\rho}-1)) + \frac{\frac{1}{2}\left(1-\frac{1}{\Delta(k)}\right)}{\int_0^1 [W(r)]^2 dr} \quad . \tag{32}$$

As the asymptotic distribution is shifted to the right if we linearly interpolate the gaps in the data, the 5% and 10% critical values will be less negative than for the usual DF statistic. The corresponding t-statistic is :

$$t_\rho = \frac{(\tilde{\rho}-1)}{[\tilde{\sigma}^2/\Sigma x_{t-1}^{/\,2}]^{\frac{1}{2}}}$$

$$where \;\; \tilde{\sigma}^2 = \frac{1}{T}\sum_{k=2}^{N} (x_t^{/} - \tilde{\rho} x_{t-1}^{/})^2 \quad . \tag{33}$$

So under the null hypothesis,

$$t_\rho = \frac{T(\tilde{\rho}-1)}{[\frac{1}{T}\Sigma a_t^{/2} / \frac{1}{T^2}\Sigma x_{t-1}^{/\,2}]^{\frac{1}{2}}} \quad . \tag{34}$$

From Phillips (1987), $1/T \; \Sigma a'^2_t \Rightarrow \sigma_5^2$ and $1/T^2 \; \Sigma x'^2_{t-1} \Rightarrow \sigma_6^2 \int_0^1 W(r)^2 dr$, implying :

$$t_\rho \Rightarrow \frac{\frac{1}{2}\left(\frac{\sigma_6}{\sigma_5}\right)[W(1)^2 - \sigma_5^2/\sigma_6^2]}{[\int_0^1 W(r)^2 dr]^{\frac{1}{2}}} \quad . \tag{35}$$

Since $\sigma_6^2 = \sigma_4^2$ and $\sigma_5^2 = \sigma_3^2(\Delta(k))^{-1}$ we have,

$$t_\rho \Rightarrow \frac{\frac{1}{2}\left(\frac{\sigma_4}{\sigma_3}\right)[\Delta(k)]^{-\frac{1}{2}}\left[W(1)^2-\left(\frac{\sigma_3^{\;2}}{\sigma_4^{\;2}}\right)(\Delta(k))^{-1}\right]}{\left[\int_0^1 W(r)^2 dr\right]^{\frac{1}{2}}} \quad . \qquad \textbf{(36)}$$

Finally, in the iid case, the limiting distribution for the t-statistic will be :

$$t_\rho \Rightarrow \frac{1}{\sqrt{\Delta(k)}}(AD(t_\rho)) + \frac{\frac{1}{2}\left(1-\frac{1}{\Delta(k)}\right)}{\left[\int_0^1 W(r)^2 dr\right]^{\frac{1}{2}}} \quad , \qquad \textbf{(37)}$$

where $AD(t_\rho)$ is the asymptotic distribution of the usual t-statistic shown in (22). We see that the limiting distribution for this t-statistic is to the right of that for the usual t-statistic. The scale decreases as well, causing the 5% and 10% critical values to be less negative than usual. Finally, the movement of this distribution to the right should increase with k.

Shin and Sarkar (1993) showed that the limit distributions of the normalized OLS estimator for ρ, and for the corresponding t-statistic, were not affected by their methods of dealing with the gaps. However, we see that filling the gaps by linear interpolation *does* affect these asymptotic distributions for the no-drift/no-trend case. So, the following asymptotic distributions will hold for the drift/no-trend and drift/trend DF regressions when dealing with gaps in the data via linear interpolation.

4.2 Drift/No-Trend

The limit distribution of the normalized OLS estimator, as in Section 2, is:

$$T(\hat{\rho}-1) \Rightarrow \frac{\frac{1}{2}\{[W(1)]^2-1\} - W(1)\cdot\int_0^1 W(r)dr}{\int_0^1 [W(r)]^2 dr - \left[\int_0^1 W(r)dr\right]^2} \quad . \qquad \textbf{(38)}$$

Shin and Sarkar note that approaches (ii) and (iii) above to deal with missing data leave this result unaltered. However, linearly interpolating to deal with the gaps :

$$T(\tilde{\rho}-1)\Rightarrow\frac{\frac{1}{2}\{[W(1)]^2-\sigma_5^{\;2}/\sigma_6^{\;2}\}-W(1)\int_0^1 W(r)dr}{\int_0^1[W(r)]^2dr-[\int_0^1 W(r)dr]^2} \quad . \tag{39}$$

As $\sigma_6^{\;2}=\sigma_4^{\;2}$ and $\sigma_5^{\;2}=\sigma_3^{\;2}(\Delta(k))^{-1}$,

$$T(\tilde{\rho}-1)\Rightarrow\frac{\frac{1}{2}\{[W(1)]^2-\frac{\sigma_3^{\;2}}{\sigma_4^{\;2}}\Delta(k)^{-1}\}-W(1)\int_0^1 W(r)dr}{\int_0^1[W(r)]^2dr-[\int_0^1 W(r)dr]^2} \quad , \tag{40}$$

or,

$$T(\tilde{\rho}-1)\Rightarrow\frac{\frac{1}{2}\{[W(1)]^2-\sigma_3^2/\sigma_4^2\}-W(1)\int_0^1 W(r)dr\ +\frac{1}{2}[(\sigma_3^2/\sigma_4^2)(1-\Delta(k)^{-1})]}{\int_0^1[W(r)]^2dr-[\int_0^1 W(r)dr]^2} \quad . \tag{41}$$

That is,

$$T(\tilde{\rho}-1)\Rightarrow AD(T(\hat{\rho}-1))+\frac{\frac{1}{2}[(\sigma_3^{\;2}/\sigma_4^{\;2})(1-\Delta(k)^{-1})]}{\int_0^1[W(r)]^2dr-[\int_0^1 W(r)dr]^2} \quad , \tag{42}$$

where AD(T($\hat{\rho}$ -1)) is the asymptotic distribution from (38). Finally, if the errors are iid,

$$T(\tilde{\rho}-1) \Rightarrow \boldsymbol{AD}(T(\hat{\rho}-1)) + \frac{\frac{1}{2}(1-\Delta(k)^{-1})}{\int_0^1 [W(r)]^2 dr - [\int_0^1 W(r)dr]^2} \quad . \tag{43}$$

The limiting distribution of the t-statistic in the iid case, also from Section 2, is :

$$t_\rho \Rightarrow \frac{\frac{1}{2}\{[W(1)]^2-1\} - W(1)\cdot\int_0^1 W(r)dr}{\{\int_0^1 [W(r)]^2 dr - [\int_0^1 W(r)dr]^2\}^{1/2}} \quad . \tag{44}$$

Using linear interpolation alters the limiting distribution in the following way:

$$t_\rho \Rightarrow \frac{\frac{\sigma_6}{\sigma_5}\{\frac{1}{2}[[W(1)]^2 - \sigma_5^2/\sigma_6^2] - W(1)\int_0^1 W(r)dr\}}{[\int_0^1 [W(r)^2]dr - [\int_0^1 W(r)dr]^2]^{1/2}} \quad . \tag{45}$$

As $\sigma_6^2 = \sigma_4^2$ and $\sigma_5^2 = \sigma_3^2(\Delta(k))^{-1}$,

$$t_\rho \Rightarrow \frac{\frac{\sigma_4}{\sigma_3}[\Delta(k)]^{-1/2}\{\frac{1}{2}[[W(1)]^2 - (\sigma_3^2/\sigma_4^2)\Delta(k)^{-1}] - W(1)\int_0^1 W(r)dr\}}{[\int_0^1 [W(r)^2]dr - [\int_0^1 W(r)dr]^2]^{1/2}} \quad , \tag{46}$$

and if errors are iid then:

$$t_\rho \Rightarrow \Delta(k)^{-1/2}(\boldsymbol{AD}(t_\rho)) + \frac{\frac{1}{2}(1-\Delta(k)^{-1})}{[\int_0^1 [W(r)]^2 dr - [\int_0^1 W(r)dr]^2]^{1/2}} \quad , \tag{47}$$

where AD(t_ρ) is the asymptotic distribution of the usual t-statistic reported in (44).

4.3 Drift/Trend

The limit distribution of the normalized OLS estimator $\hat{\rho}$ in the iid case is shown in equation (9), namely :

$$T(\hat{\rho}-1) \Rightarrow \frac{\frac{1}{2}([W(1)]^2-1)-W(1)A}{D} \quad (48)$$
$$A=[4\int W(r)dr-6\int rW(r)dr]-[W(1)-\int W(r)dr][12\int rW(r)dr-6\int W(r)dr]$$
$$D=[\int[W(r)]^2dr+12\int W(r)dr\int rW(r)dr-4[\int W(r)dr]^2-12[\int rW(r)dr]^2]$$

where all integrals are over the range [0,1].

Again, from Shin and Sarkar, this result also holds if methods (ii) or (iii) above are used to deal with the missing data. However, using linear interpolation :

$$T(\tilde{\rho}-1) \Rightarrow \frac{\{\frac{1}{2}[W(1)]^2-\Delta(k)^{-1}\}-W(1)A}{D} \quad , \quad (49)$$

where, to simplify the exposition, we have presented only the iid result in (49).
Similarly, in the iid case, the usual limiting distribution of the t-ratio, from (9), is :

$$t_\rho \Rightarrow \frac{\frac{1}{2}([W(1)]^2-1)-W(1)A}{D^{1/2}} \quad . \quad (50)$$

However, under linear interpolation:

$$t_\rho \Rightarrow \frac{\Delta(k)^{-1/2}\{\frac{1}{2}[[W(1)]^2-\Delta(k)^{-1}]-W(1)A\}}{D^{1/2}} \quad , \quad (51)$$

where A and D are defined in (48).

4.4 Augmented Dickey-Fuller Tests

The above discussion is all in terms of “simple” (rather than “augmented”) DF tests. The ADF and DF "t-statistics" have the same limit distributions, as long as we choose the augmentation level to grow with T (Banerjee *et al.* (1993, p106) and Said and Dickey (1984)). So, asymptotically, augmenting does not affect the above results relating to missing data. However, augmenting *does* affect the finite sample properties of the tests (*e.g.*, Schwert (1989) and Dods and Giles (1995)). Therefore, the ADF test is considered separately from the DF test in the simulation experiment described in Section 5.

5. A SIMULATION EXPERIMENT

In practice, the finite-sample properties of all three procedures for dealing with the missing data are of considerable interest. We have used Monte Carlo simulation to compute the empirical sizes (in the sense described earlier) of the modified DF tests. These sizes were then compared with their *nominal* sizes based on the 5% and 10% critical values of MacKinnon (1991), which assume no breaks in the data. *Size-adjusted* powers were obtained as follows. Using the simulation results associated with empirical sizes above, the the true critical value needed to cut off exactly 5% or 10% in the left tail of each null distribution was found by iteration. Then, the models were simulated under the alternative hypothesis, for various stationary values of ρ, and using this exact critical value (rather than the nominal MacKinnon critical value), rejection rates were computed for the null hypothesis. When the DGP contained MA errors, as in equation (53) below, Monte Carlo simulations were run to obtain the replacement critical values in order to develop power curves for *each value* of θ.

The interest in such size-adjusted power curves, even though the typical practitioner may not adjust the critical values this way, is that they can be compared with one another directly for the purposes of assessing power rankings. In addition, *unadjusted* power curves were obtained by using MacKinnon’s critical values under the alternative hypothesis, as would most applied researchers in practice. These curves represent *raw rejection rates* of the I(1) null when it is false to some degree, and care must be taking in comparing them directly[3].

Modified ADF tests for a unit root are also considered here. The level of augmentation chosen followed evidence from Dods and Giles (1995), who showed that for sample sizes up to 100 observations, assigning p = 4 is an effective strategy in terms of low size-distortion. As we consider samples only up to this size, this same value of p is used to simplify the following analysis. (There is clearly scope for further research based on other choices of p). The three usual ADF regression equations are:

$$\Delta Y_t = (\rho - 1)Y_{t-1} + \sum_{i=1}^{4} \gamma_i \Delta Y_{t-i} + u_t'$$
$$\Delta Y_t = \mu + (\rho - 1)Y_{t-1} + \sum_{i=1}^{4} \gamma_i \Delta Y_{t-i} + u_t'' \qquad (52)$$
$$\Delta Y_t = \mu + \beta t + (\rho - 1)Y_{t-1} + \sum_{i=1}^{4} \gamma_i \Delta Y_{t-i} + u_t''' \quad ,$$

where it is assumed that u_t', u_t'', and u_t''' are iid N(0,1).

The time-series considered in our study follow the A-B sampling scheme used by Shin and Sarkar. That is, there are A observed values followed by B missing values, repeated m times over the sample, so the total number of available observations is mA. Under this scheme observations are defined by the k (A×1) vectors Y_0, Y_1,...,$Y_{(k-1)}$, where $Y'_h = (y_{h(A+B)+1},\ldots,y_{h(A+B)+A})$ for h = 0,1,...,(k-1). Two forms of missing observations are examined, and these could be motivated by stock (rather than flow) data are generated monthly but only observed quarterly (A = 1 and B = 2), or generated quarterly but observed annually (A = 1 and B = 3). Three different values of m (10, 20, 30) were used, but for the ADF tests only m = 20, 30 were used. The total numbers of observations, including gaps, for each value of m were 28, 58, and 88 for the A = 1, B = 2 case; and 37, 77, and 117 for the A = 1, B = 3 case. The experiment was replicated thousand replications all combinations of m = 10, 20, 30; A-B = 1-2 , 1-3; ρ = 1.0 (0.05) 0.70; and nominal significance levels of 5% and 10%. The DGP with moving average errors was[4] :

$$Y_t = \rho Y_{t-1} + (\epsilon_t - \theta \epsilon_{t-1}) \quad , \qquad (53)$$

where θ = 0 ±0.25, ±0.50, ±0.75.

6. RESULTS[5]

6.1 Regular Dickey-Fuller Regressions

The series Y_t was generated as :

$$Y_t = \rho Y_{t-1} + \epsilon_t \quad . \tag{54}$$

where $Y_0 = 0$, $\epsilon_t \sim$ iid N(0,1). We then considered the following DF regressions:

$$\begin{aligned} \Delta Y_t &= (\rho - 1) Y_{t-1} + \epsilon_t \\ \Delta Y_t &= \mu + (\rho - 1) Y_{t-1} + \epsilon_t^{*} \\ \Delta Y_t &= \mu + \beta t + (\rho - 1) Y_{t-1} + \epsilon_t^{**} \quad , \end{aligned} \tag{55}$$

where the error terms are assumed to be i.i.d. with constant mean and variance. For all values of m (the number of observed values), for both nominal significance levels, for both A-B sampling schemes, and for all three DF regressions, less “size distortion” (*i.e.*, the difference between pre-assigned significance level and the empirical rejection rate under the null) emerged when the series was "closed", as opposed to when the gaps were filled in with the last observed value. Tables 1 and 2 illustrate this. However, these results were less clear when the gaps were filled by linear interpolation. The *amount* of size distortion depended on the form of the DF regression. When the gaps were ignored, the "no-drift/no-trend" and "drift/trend" DF tests had less size distortion than their linear interpolation counterparts, for all values of m, and both significance levels, and both A-B sampling schemes. When ignoring the gaps, the "drift/no-trend" tests sometimes had more size distortion than did their linear interpolation counterparts, for all scenarios examined, as is shown in Table 3 for m = 20.

When comparing the two methods of filling in the gaps, the sample size determined which method had less size distortion. By filling in the gaps by linear interpolation for m = 10, 20, for both nominal significance levels, and for both sampling schemes, less size distortion was evident than when the last observed value was used. However, this advantage disappeared when m = 30.

A comparison of size-adjusted powers yielded similar results: ignoring the gaps is a more powerful strategy than the other alternatives. Figure 1 and Tables 4 and 5 illustrate this for m = 30 and A-B = 1-2, at the 10% significance level. This power gain was especially pronounced when compared with the linear interpolation strategy. The power curves resulting from closing the series increase as ρ decreases and were always greater than their counterparts for the linear interpolation case, for both A-B sampling schemes, and for all DF regressions. In fact, the power curves for the linear interpolation method generally *decreased* with ρ, for all values of m and for both A-B sampling schemes. Exceptions included the "no-drift/no-trend" DF regressions for m = 10.

Filling the gaps with the last known value produced power curves that increased as ρ decreased for m = 20, 30, and for both A-B sampling schemes. However, these powers were dominated by those associated with "closing" the gaps in the data. In addition, the power curves associated with tests based on the "no-drift/no-trend" DF regressions were exactly the same under both of these approaches. This did not occur when the linear interpolation method was used. This interesting result has been investigated further, but without a complete resolution. We have established that it does *not* arise from an algebraic equivalence between the (adjusted) test statistics. It appears to be a distributional result, and we are examining this issue further in related work.

Examining the power curves we see that filling in the gaps with the last known observation produces unit root tests which are *far more* powerful in finite samples than those based on the linear interpolation approach, which can result in a biased test. Apparently this arises because the latter method introduces non-independent values to “fill the gaps”. For all values of m, for both significance levels, for all DF regressions, and for both A-B sampling schemes, the previous observed value method of filling the gaps in the series resulted in the highest power curves. Figure 1 shows the corresponding power curves for m = 30, for A-B = 1-2, and for the "drift/trend" DF regression, at the 10% significance level.

6.2 MA Errors: Dickey-Fuller Tests

This section describes the nominal sizes and powers for all three ways of dealing with the gaps when the DGP under MA errors is as given in Section 5, and $\theta = 0.0$ or 0.50. For the other values of θ, the linear interpolation procedure was dropped due to the low power of the tests.

Considering first the situation when $\theta \geq 0$, the specification of the DF regression was an important determinant of the method exhibiting the least amount of size distortion for m = 10, 20. When the DF regression was specified with either drift/no-trend, or drift/trend, less size distortion was evident in the closed series as compared with the series where the gaps were filled with the last known observation. On the other hand, replacing the gaps with the last known value generated the least amount of size distortion when the no- drift/no-trend DF regression was used, for both A-B sampling schemes and both significance levels. However, for m = 30, filling the gaps produced *less* size distortion than using the closed series for all DF regressions. Linear interpolation generally produced tests with *less* size distortion than those corresponding to closing the series or using the previous known observation to replace the gaps. For both sampling schemes, both significance levels, and m = 20, 30, linear interpolation minimized the degree of size distortion.

Closing the series produced unit root tests which were *more powerful* than those obtained by filling the gaps. As m increased or θ decreased, the differential between the power curves decreased. Figures 2 and 3 show the power curves for representative drift/trend cases.

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When size-adjusted powers were calculated, the linear interpolation method was *far* less powerful than closing the series. Table 6 shows the size-adjusted powers for A-B = 1-2 sampling scheme and m = 30 at the 10% significance level for the closed series and the series filled by linear interpolation. Filling in the series with the last known value and ignoring the gaps produced tests which were comparable in power, with the linear interpolation method being quite inferior.

As expected, $\theta = 0.75$ produced a large amount of size distortion for both the closed and full series and this distortion increased with the value of m. Filling the gaps resulted in *less* size distortion than did using the closed series for all values of m, for both sampling schemes, for all DF regressions, and for both significance levels. Table 7, for example, reports the sizes when m = 10 and A-B = 1-2 at the 10% significance level. The unit root tests were *more powerful* when the gaps were ignored instead of replaced. Figure 4 shows the power curves for $\theta = 0.75$, m = 30, A-B = 1-2, and the drift/trend DF regression, at the 10% significance level. However, the unit root tests were not as powerful as θ approached one. Figure 5 shows representative power curves for various $\theta \geq 0$.

With $\theta < 0$, closing the series produced *less* size distortion as compared to filling the gaps in the data, and this distortion increased as drift and/or trend terms were added to the DF regressions, but did not vary systematically with θ. Again, the powers of the unit root tests were *greater* when the gaps in the time-series were ignored.

The unit root tests remained *more* powerful when the gaps were ignored, as compared to filling the gaps in some way, as is illustrated in Figure 6. Further, the powers of the unit root tests are generally greater for positive θ than for their counterparts when θ is negative.

6.3 MA Errors: Augmented Dickey-Fuller Tests

The DGP remained the same as in Section 6.2, but now the usual DF regressions were augmented in the same way as Said and Dickey (1981), and as reported in equation (52) above, with an augmentation level of p = 4 following the eveidence of Dods and Giles (1995). For illustrative purposes, we discuss here just the results relating to $\theta = 0$ and $\theta = \pm 0.50$.

For $\theta \geq 0$ and sampling scheme A-B =1-3 (with m = 20 and 30, and the two significance levels) *less* size distortion arose when the gaps were replaced with the last known observation, as opposed to closing the series. However, for sampling scheme, A-B = 1-2, both values of m, and both significance levels, the opposite was true. (This relationship did not hold for the no-drift/no-trend DF regression for both values of m, and for both significance levels.) The converse was true for $\theta < 0$, and all forms of the ADF regression.

In the case of size-adjusted power, augmenting the DF regressions produced curious results: filling the gaps with the last known observation produced *more* powerful unit root tests in comparison with the powers of the unit root tests obtained by ignoring the gaps. This result did *not* depend on the degree or sign of the MA effect, as can be seen from Figures 7, 8 and 9, and neither did it depend on the inclusion or exclusion of drift or trend terms in the ADF regressions.

7. CONCLUSIONS

In this study we have considered the asymptotic and finite-sample distributions of the normalized OLS estimator for ρ, and the corresponding "t-statistic", for testing the null of a unit root in time-series data with missing observations. Shin and Sarkar (1993) give the limiting distributions associated with a set of data with no missing observations, a time-series where the gaps are ignored (*i.e.,* the series was closed), and a time-series where the gaps are replaced with the last available observation. They show that replacing the gaps with the last observation, or simply ignoring the gaps, does not alter the usual asymptotic results associated with such Dickey-Fuller statistics. We reproduce these results in this paper.

We have extended their results by proposing a linear interpolation method for dealing with the gaps in the data. We prove that the limiting distribution of the normalized OLS estimator of ρ, $T(\tilde{\boldsymbol{\rho}}-1)$, includes an adjustment factor which results in critical values that are less negative than for the usual DF statistic. Further, the resulting asymptotic distribution for the DF t-statistic also includes an adjustment factor which causes the critical values to be less negative than usual. It is important to note that *all* of these asymptotic results are invariant to the number or pattern of missing data-points in the series.

We have investigated the finite-sample properties of these tests via a simulation experiment which uses the repetitive "A-B sampling scheme" to create gaps in the time-series. While Shin and Sarkar addressed only size-unadjusted powers, our study also considers size-adjusted powers, thus facilitating proper comparisons. We have also extended the earlier work by allowing for DGP's with moving-average errors, and the "augmented" DF tests. Although the level of size distortion among the methods of dealing with the gaps varied throughout the study, closing the series produced the best power curves under all circumstances.

The regular DF unit root tests were always *more* powerful when the gaps were ignored, as compared with the two alternatives. When replacing the gaps, using the last available observation produced more powerful unit root tests than those calculated when using linear interpolation to fill the gaps in the data.

With MA errors imposed in the DGP and using regular DF regressions, closing the series proved to be *more* powerful than replacing the gaps with the last available observation. When using the augmented DF procedure, the power of the unit root test was *superior* when the gaps were replaced instead of ignored. As a result, this finding would suggest filling the gaps with the last known observation when augmenting the DF regressions to combat the existence of MA errors.

It is clear from this study that when testing for a unit root using regular Dickey-Fuller type procedures in time-series data with missing observations, ignoring the gaps (*i.e.*, closing the series) will yield the best results. However, when using the augmented DF procedure to combat MA errors in time-series data with missing observations, filling the gaps in the data with the last known observation yields more powerful unit root tests, as compared to ignoring the gaps. Furthermore, regardless of the method used to deal with the gaps, be it ignoring or replacing with the last known observation, we can still use the regular DF critical values because the estimator for ρ in the resulting series has the same asymptotic distribution as the estimator for ρ in a series with no missing observations.

Although these results provide some useful prescriptions for applied researchers who wish to use Dickey-Fuller unit root tests with incomplete or infrequently observed time-series data, there is scope for further research. First, it must be re-emphasised that we have considered only DGP's which exclude both drift and trend, and the above prescriptions may very well change in more general settings. Further, Shin and Sarkar's (1994a) study of Hall's instrumental variable approach could be extended to include size-adjusted power; and other ways of choosing the augmentation levels for the augmented Dickey-Fuller unit root tests could be explored. We explored the possibility of using a dummy-variables approach (in the spirit of Perron (1989)) to DF testing with missing data, and the discouraging results are reported by Ryan (1996). Other unit root tests could be evaluated in the context of missing data. For instance, the Durbin-Watson testing approach of Hisamatsu and Maekawa (1994), seasonal unit root tests (*eg.*, Hylleberg *et al.* (1990)), and other unit root tests constructed so that the null hypothesis is that the series is stationary and the alternative is that it is I(1) (*e.g.*, Kwiatkowski *et al.* (1992)) could all be analyzed. Although we have not considered the Phillips-Perron (1988) test in this study, it would be worthy of investigation in the context of missing observations and their mixing-type heterogeneity of the errors. Finally, there is the important problem of the effects of missing data on tests for cointegration between different time-series.

As we have emphasized in introducing the work in the paper, missing or infrequent observations are a fact of life when working with economic time-series data, and it will be important to consider the implications of such contaminations for the full suite of tests that underly modern time-series econometrics.

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NOTES

1. See Harvey (1989, p.304 and pp.406-407).

2. Here, and throughout the rest of this paper, "empirical size" (or just "size") will imply the proportion of times the null is rejected when it is true, regardless of any other contamination effects in the overall model. "Power" will be defined analogously when the null is false.

3. Note, however, that if one test has *lower* empirical size than a second test, and also has *higher* unadjusted power than the second test (under the same conditions), then the first test is unambiguously *more powerful* than is the second, under those conditions.

4. We have limited our study to the situation where the DGP excludes drift or trend terms. This accords, of course, with the standard Dickey-Fuller analysis, but is rather restrictive in practice. We are grateful to Tom Fomby for correctly pointing out that if the maintained hypothesis allowed for drift and trend, one might expect a better performance from both the "closing the gaps" and "interpolation methods, especially as the gap lengths increase. This caveat should be borne in mind when interpreting the following results.

5. See Ryan (1996) for a more comprehensive set of results from this study.

REFERENCES

Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle, in B. Petrov and F. Csake, eds, *2nd International Symposium on Information Theory*, Budapest: Akademiai Kiado.

Bhargava, A. (1989). Missing observations and the use of the Durbin-Watson statistic, *Biometrika* 76, 828-831.

Box, G. E. P. and G. M. Jenkins (1976). *Time Series Analysis: Forecasting and Control*, San Francisco: Holden-Day.

Davies, R. B. (1980). The distribution of a linear combination of χ^2 random variables, algorithm AS 155, *Applied Statistics* 29, 323-333.

Dickey, D. A. and W. Fuller (1979). Distribution of the estimators for autoregressive time series with a unit root, *Journal of the American Statistical Association* 74, 427-431.

Dickey, D. A. and W. Fuller (1981). Likelihood ratio statistics for autoregressive time series with a unit root, *Econometrica* 49, 1057-1072.

Dickey, D. A. and S. E. Said (1981). Testing ARIMA (p,1,q) versus ARIMA (p+1,q), *Proceedings of the Business and Economic Statistics Section, American Statistical Association*, 318-322.

Dods, J. L. and D. E. A. Giles (1995). Alternative strategies for 'augmenting'the Dickey-Fuller test: Size-robustness in the face of pre-testing, *Journal of Statistical Computation and Simulation* 53, 243-258.

Fuller, W. (1976). *Introduction to Statistical Time Series*, New York: Wiley.

Hall, A. (1989) Testing for a unit root in the presence of moving average errors, *Biometrika* 76, 49-56.

Hall, A. (1992). Testing for a unit root in time series using instrumental variable estimators with pretest data based model selection, *Journal of Econometrics* 54, 223-250.

Hall, A. (1994). Testing for a unit root in time series with pretest data based model selection, *Journal of Business and Economic Statistics* 12, 461-470.

Hamilton, J. D. (1994). *Time Series Analysis*, Princeton: Princeton University Press.

Hannan, E. and B. Quinn (1979). The determination of the order of an autoregression, *Journal of Royal Statistical Society* B 41, 190-195.

Harvey, A. C. (1989). *Forecasting, Structural Time Series and the Kalman Filter*, Cambridge: Cambridge University Press.

Harvey, A. C. and R. G. Pierse (1984). Estimating missing observations in economic time series, *Journal of American Statistical Association* 79, 125-131.

Hisamatsu, H. and K. Maekawa (1994). The distribution of the Durbin-Watson statistic in integrated and near-integrated models, *Journal of Econometrics* 61, 367-382.

Hylleberg, S., R. F. Engle, C. W. J. Granger, and B. S. Yoo (1990). Seasonal integration and co-integration, *Journal of Econometrics* 44, 215-228.

Kohn, R. E. and C. F. Ansley (1986). Estimation, prediction and interpolation for ARIMA models with missing data, *Journal of the American Statistical Association* 81, 751-761.

Kwiatkowski, D., P. C. B. Phillips, P. Schmidt, and Y. Shin (1992). Testing the null hypothesis of stationarity against the alternative of a unit root: How sure are we that economic time series have a unit root?, *Journal of Econometrics* 54, 159-178.

Little, R. J. A. and D. B. Rubin (1983). On jointly estimation parameters and missing data, *Journal of the American Statistical Association* 37, 218-220.

Little, R. J. A. and D. B. Rubin (1987). *Statistical Analysis with Missing Data*, New York: Wiley.

Newey, W. and K. West (1987). A simple positive semi-definite, heteroscedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703-708.

MacKinnon, J. G. (1991). Critical values for cointegration tests, in R. F. Engle and C. W. J. Granger (eds.), *Long-Run Economic Relationships*, Oxford: Oxford University Press, 267-276.

Pantula, S. and A. Hall (1991). Testing for unit roots in autoregressive moving average models: An intrumental variable approach, *Journal of Econometrics* 48, 325-353.

Perron, P. (1989). The great crash, the oil price shock and the unit root hypothesis, *Econometrica* 57, 1361-1401.

Phillips, P. C. B. (1986). Understanding spurious regressions in econometrics, *Journal of Econometrics* 33, 311-340.

Phillips, P. C. B. (1987). Time series regression with unit roots, *Econometrica* 55, 277-302.

Phillips, P. C. B. and P. Perron (1988). Testing for a unit root in time series regression, *Biometrika* 75, 335-346.

Ryan, K. F. (1996). Testing for unit roots in time series data with missing observations, Unpublished M.A. Extended Essay, University of Victoria.

Said, S. E. and D. A. Dickey (1984). Testing for unit roots in autoregressive moving average models of unknown order, *Biometrika* 71, 599-607.

Savin, N. E. and K. J. White (1978). Testing for autocorrelation with missing observations, *Econometrica* 46, 59-67.

Schwert, G. W. (1989). Tests for unit roots: a Monte Carlo investigation, *Journal of Business and Economic Statistics* 7, 147-159.

SHAZAM (1997). *Shazam Econometrics Computer Program: User's Reference Manual Version 8.0*, New York: McGraw-Hill.

Shin, D. W. and S. Sarkar (1993). Testing for a unit root in an AR(1) time series using irregularly observed data, Unpublished manuscript, Oklahoma State University.

Shin, D. W. and S. Sarkar (1994a). Unit root tests for ARIMA (0,1,q) models with irregularly observed samples, *Statistics and probability letters* 19, 188-194.

Shin, D. W. and S. Sarkar (1994b). Likelihood ratio type unit root tests for AR(1) models with nonconsecutive observations, *Communications in Statistics: Theory and Methods* 23, 1387-1397.

Table 1: Unadjusted Powers : DGP Equation (54)

(A-B = 1-3; m = 10; θ = 0.0, 5% significance level)

ρ	G1	G2	G3	P1	P2	P3
1.00	0.0524	0.0206	0.0268	0.0276	0.0120	0.0108
0.95	0.1195	0.0399	0.0279	0.0611	0.0120	0.0076
0.90	0.2199	0.0666	0.0387	0.1129	0.0107	0.0077
0.85	0.3446	0.1000	0.0556	0.1950	0.0102	0.0076
0.80	0.4616	0.1369	0.0765	0.2882	0.0117	0.0077
0.75	0.5634	0.1810	0.1019	0.3783	0.0131	0.0073
0.70	0.6394	0.2275	0.1264	0.4566	0.0173	0.0068

Note: G1 - No drift, no trend, gaps closed
G2 - Drift, no trend, gaps closed
G3 - Drift, trend, gaps closed
P1 - No drift, no trend, previous value used
P2 - Drift, no trend, previous value used
P3 - Drift, trend, previous value used

Table 2: Unadjusted Powers : DGP Equation (54)

(A-B = 1-3; m = 30; θ = 0.0; 10% significance level)

ρ	G1	G2	G3	P1	P2	P3
1.00	0.0969	0.0766	0.0757	0.0879	0.0754	0.0640
0.95	0.6051	0.2400	0.1597	0.5668	0.1841	0.1037
0.90	0.9406	0.5528	0.3564	0.9269	0.4510	0.2278
0.85	0.9931	0.8628	0.6661	0.9732	0.7996	0.5620
0.80	0.9986	0.9354	0.7831	0.9982	0.8918	0.6276
0.75	0.9995	0.9804	0.8964	0.9995	0.9586	0.7760
0.70	0.9998	0.9922	0.9515	0.9998	0.9848	0.8709

Note: G1 - No drift, no trend, gaps closed
G2 - Drift, no trend, gaps closed
G3 - Drift, trend, gaps closed
P1 - No drift, no trend, previous value used
P2 - Drift, no trend, previous value used
P3 - Drift, trend, previous value used

Table 3: Unadjusted Powers : DGP Equation (54)

(A-B = 1-3; m = 20; θ = 0.0; 5% significance level)

ρ	G1	G2	G3	L1	L2	L3
1.00	0.0561	0.0286	0.0318	0.0001	0.0521	0.0202
0.95	0.2344	0.0823	0.0515	0.0003	0.0026	0.0080
0.90	0.5341	0.1842	0.1051	0.0027	0.0000	0.0020
0.85	0.7814	0.3355	0.1908	0.0103	0.0000	0.0013
0.80	0.9091	0.4900	0.2979	0.0203	0.0000	0.0007
0.75	0.9589	0.6320	0.4095	0.0600	0.0001	0.0003
0.70	0.9801	0.7370	0.5093	0.1048	0.0003	0.0002

Note: G1 - No drift, no trend, gaps closed
G2 - Drift, no trend, gaps closed
G3 - Drift, trend, gaps closed
L1 - No drift, no trend, linear interpolation used
L2 - Drift, no trend, linear interpolation used
L3 - Drift, trend, linear interpolation used

Table 4: Size-Adjusted Powers : DGP Equation (54)

(A-B = 1-2; m = 30; θ = 0.0; 10% significance level)

ρ	G1	G2	G3	P1	P2	P3
1.00	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
0.95	0.4432	0.2338	0.1609	0.4432	0.2013	0.1340
0.90	0.8202	0.4707	0.3053	0.8202	0.4101	0.2412
0.85	0.9628	0.7148	0.4885	0.9628	0.6502	0.3989
0.80	0.9937	0.8752	0.6795	0.9937	0.8294	0.5751
0.75	0.9986	0.9516	0.8159	0.9986	0.9296	0.7315
0.70	0.9994	0.9809	0.9050	0.9994	0.9725	0.8420

Note: G1 - No drift, no trend, gaps closed
G2 - Drift, no trend, gaps closed
G3 - Drift, trend, gaps closed
P1 - No drift, no trend, previous value used
P2 - Drift, no trend, previous value used
P3 - Drift, trend, previous value used

Table 5: Size-Adjusted Powers : DGP Equation (54)

(A-B = 1-2; m = 30; θ = 0.0; 10% significance level)

ρ	G1	G2	G3	L1	L2	L3
1.00	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
0.95	0.4432	0.2338	0.1609	0.4398	0.0190	0.0715
0.90	0.8202	0.4701	0.3053	0.7718	0.0286	0.0662
0.85	0.9628	0.7148	0.4885	0.9109	0.0712	0.1000
0.80	0.9937	0.8752	0.6795	0.9596	0.1403	0.1624
0.75	0.9986	0.9516	0.8159	0.9790	0.2483	0.2501
0.70	0.9994	0.9809	0.9050	0.9892	0.3774	0.3584

Note: G1 - No drift, no trend, gaps closed
G2 - Drift, no trend, gaps closed
G3 - Drift, trend, gaps closed
L1 - No drift, no trend, linear interpolation used
L2 - Drift, no trend, linear interpolation used
L3 - Drift, trend, linear interpolation used

Table 6: Size-Adjusted Powers : DGP Equation (53)

(A-B = 1-2; m = 30; θ = 0.50; 10% significance level)

ρ	G1	G2	G3	L1	L2	L3
1.00	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
0.95	0.4424	0.2728	0.1632	0.4473	0.0854	0.0929
0.90	0.8097	0.5117	0.2891	0.7912	0.1626	0.1233
0.85	0.9553	0.7307	0.4464	0.9313	0.2921	0.1897
0.80	0.9917	0.8625	0.5974	0.9717	0.4392	0.2749
0.75	0.9973	0.9306	0.7110	0.9864	0.5666	0.3717
0.70	0.9989	0.9617	0.7955	0.9906	0.6636	0.4604

Note: G1 - No drift, no trend, gaps closed
G2 - Drift, no trend, gaps closed
G3 - Drift, trend, gaps closed
L1 - No drift, no trend, linear interpolation used
L2 - Drift, no trend, linear interpolation used
L3 - Drift, trend, linear interpolation used

Table 7: Unadjusted Powers : DGP Equation (53)

(A-B = 1-2; m = 10; θ = 0.75; 10% significance level)

ρ	G1	G2	G3	P1	P2	P3
1.00	0.4486	0.4233	0.3583	0.3809	0.2631	0.1229
0.95	0.6762	0.4695	0.3686	0.5962	0.2363	0.1126
0.90	0.8190	0.4944	0.3885	0.7429	0.2243	0.0989
0.85	0.8847	0.5296	0.4048	0.8284	0.2364	0.0841
0.80	0.9169	0.5642	0.4225	0.8714	0.2581	0.0724
0.75	0.9336	0.5941	0.4345	0.8935	0.2742	0.0655
0.70	0.9436	0.6185	0.4446	0.9056	0.2868	0.0601

Note: G1 - No drift, no trend, gaps closed
G2 - Drift, no trend, gaps closed
G3 - Drift, trend, gaps closed
P1 - No drift, no trend, previous value used
P2 - Drift, no trend, previous value used
P3 - Drift, trend, previous value used