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Improved Maximum Likelihood Estimation of the Shape Parameter in the Nakagami Distribution

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Abstract

We develop and evaluate analytic and bootstrap bias-corrected maximum likelihood estimators for the shape parameter in the Nakagami distribution. This distribution is widely used in a variety of disciplines, and the corresponding estimator of its scale parameter is trivially unbiased. We find that both "corrective" and "preventive" analytic approaches to eliminating the bias, to $O(n^{-2})$, are equally, and extremely, effective and simple to implement. As a bonus, the sizeable reduction in bias comes with a small reduction in mean squared error. Overall, we prefer analytic bias corrections in the case of this estimator. This preference is based on the relative computational costs and the magnitudes of the bias reductions that can be achieved in each case. Our results are illustrated with two real-data applications, including one which provides the first application of the Nakagami distribution to data for ocean wave heights.

Keywords: Nakagami distribution, maximum likelihood estimation, bias reduction

Mathematics Subject Classification: 62E15, 62F10, 62F40, 62P10, 62P12, 62P30

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1. Introduction

The Nakagami-m distribution (or simply the Nakagami distribution) was proposed for modeling the fading of radio signals (Nakagami, 1960). There is now a vast literature relating to its application in the general area of communications engineering. For example, a search of the *IEEE Xplore digital Library* (IEEE, 2011) on 5 May 2011, based on the word "Nakagami", located 2,041 research papers. The Nakagami distribution has also been applied successfully in various other fields. For example, Sarkar *et al.* (2009, 2010) show that it performs well in the derivation of unit hydrographs, as used to estimate runoff in hydrology. Shankar *et al.* (2005) and Tsui *et al.* (2006) use the Nakagami distribution to model ultrasound data in medical imaging studies; and Kim and Latchman (2009) use this distribution in their analysis of multimedia (MPEG-2 frame) data traffic over networks. Recently, Carcolé and Sato (2009) and Nakahara and Carcolé (2010) have shown the usefulness of the Nakagami distribution for modelling high-frequency seismogram envelopes.

A random variable, X, that follows a Nakagami distribution has the density function

$$f(x) = (2/\Gamma(\mu)) \left(\frac{\mu}{\omega}\right)^{\mu} x^{2\mu-1} exp\left(-\frac{\mu}{\omega}x^2\right) ;\ x > 0 \qquad (1)$$

where μ (≥ 0.5) is the shape parameter (often given the label 'm' and referred to as the 'fading' parameter); and ω (> 0) is the scale parameter. The Nakagami distribution is closely related to various other distributions. For example, it collapses to the Rayleigh distribution when $\mu = 1$, and to the half-normal distribution when $\mu = 0.5$. For this reason, the interval $0.5 < \mu < 1$ is sometimes referred to as the "pre-Rayleigh" range for the shape parameter, while $\mu > 1$ defines the "post-Rayleigh" range. In addition, if Y is gamma-distributed with shape and scale parameters k and θ respectively, then $\sqrt{Y}$ follows a Nakagami distribution with parameters $\mu = k$ and $\omega = k\theta$. Finally, if 2μ is integer-valued and if Z follows a chi distribution with parameters 2μ, then $\sqrt{(\omega/2\mu)Z}$ is Nakagami-distributed with parameters μ and ω.

As we will see, the estimation of the scale parameter, ω, is trivial. However, considerable attention has been paid to the estimation of the shape parameter in this distribution. Many of the estimators that have been suggested are approximations, to some degree or other, to the maximum likelihood (ML) or method of moments (MOM) estimators. For example, see Cheng and Beaulieu (2001, 2002), Hadžialić *et al.* (2007). Alternative estimators have been considered and compared by Abdi and Kaveh (2000), Gaeddert and Annamalai (2004) and Beaulieu and Chen (2007), for example. In part, the use of approximations to the "natural" estimators appears to be motivated in this context by practical considerations of implementation. Specifically, practitioners in the communications engineering field often wish to avoid the need to solve the non-linear first-order conditions associated with MLE because this computation has to be undertaken many times, very rapidly, in real time. Approximations that involve "look-up tables" are computationally more convenient. Typically, this is not a consideration in other areas of application. In addition, in some cases there seems to be a lack of awareness (*e.g.*, Zhang, 2002) of the accuracy and computational ease of modern routines for evaluating the non-standard functions that enter the first-order conditions. The ML estimator (MLE) is attractive in view of its usual desirable asymptotic properties, as is exemplified by Nakahara and Carcolé (2010) in a different area of application.

In this paper, we re-visit MLE for the parameters of the Nakagami distribution, and focus on performance in small samples. From (1), the log likelihood function based on n i.i.d. sample observations is

$$l(\mu,\omega|x) = nln(2) + n\mu ln(\mu) - nln\Gamma(\mu) - n\mu ln(\omega) \qquad (2)$$
$$+(2\mu-1)\sum_{i=1}^{n} ln(x_i) - \frac{\mu}{\omega}\sum_{i=1}^{n} x_i^2$$

It follows immediately that the MLE for the scale parameter is $\hat{\omega} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$. As the second raw moment for the Nakagami distribution is just ω, it follows immediately that the MLE coincides with the MOM estimator, and is exactly unbiased.

Obtaining the MLE for the shape parameter involves profiling (2) and then solving the associated non-linear first-order condition for $\hat{\mu}$. There is no closed-form solution to this problem. However, as we show in section 3, to $O(n^{-1})$, $\hat{\mu}$ is upward-biased in finite samples. Accurate measures of the scale parameter are of paramount importance in applications of the Nakagami distribution. Accordingly, we consider various methods for measuring and correcting for the bias of its MLE. Specifically, we use two analytic techniques - one "corrective", and one "preventive" - to estimate the second-order bias of $\hat{\mu}$, and to bias-correct this estimator. We compare the performance of these modified estimators with that of another bias-corrected MLE of μ, based on the parametric bootstrap.

In the next section we discuss the analytic bias adjustment methods in general terms; and in section 3 we provide the details of their application to the MLE for the shape parameter in the Nakagami distribution. The results of a Monte Carlo simulation that evaluates the bias of $\hat{\mu}$, and compares the analytic bias-corrections with one based on the bootstrap, are reported in section 4. In section 5, we illustrate our main results with several applications involving real data-sets of modest size. Secction 6 concludes.

2. Bias-reduced maximum likelihood estimation

2.1. Definitions

Let $l(\theta)$ be the log-likelihood function where the p-dimensional vector of parameters, θ, is to be estimated using a sample of n observations. Assume that the log-likelihood function is well behaved and satisfies the usual regularity conditions (Duguét, 1937; Cramér, 1946).

The joint cumulants of $l(\theta)$ are:

$$\begin{aligned}
\kappa_{ij} &= E\left(\frac{\partial^2 l}{\partial\theta_i\theta_j}\right); \quad i,j = 1,2,\ldots,p \ . \\
\kappa_{ijl} &= E\left(\frac{\partial^3 l}{\partial\theta_i\partial\theta_j\partial\theta_l}\right); \quad i,j,l = 1,2,\ldots,p \ . \\
\kappa_{ij,l} &= E\left[\left(\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}\right)\left(\frac{\partial l}{\partial\theta_l}\right)\right]; \quad i,j,l = 1,2,\ldots,p.
\end{aligned}$$

The derivatives of these cumulants are:

$$\kappa_{il}^{(l)} = \partial\kappa_{ij}/\partial\theta_l; \ i,j,l = 1,2,\ldots,p.$$

Fisher's information matrix is $K = \{-\kappa_{ij}\}$, each element of which is $O(n)$.

2.2. A corrective approach

Cox and Snell (1968) showed that with a sample of independent data that are not necessarily identically distributed, the bias of the s^{th} element of $\hat{\theta}_s$ can be written as:

$$Bias(\hat{\theta}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} \kappa^{si}\kappa^{jl}[\frac{1}{2}\kappa_{ijl} + \kappa_{ij,l}] + O(n^{-2}); \;\; s = 1, 2, \ldots, p$$

where κ^{ij} is the $(i, j)^{th}$ element of the inverse of the information matrix, K. Furthermore, Cordeiro and Klein (1994) showed that the previous equation can be written in the following convenient form, and can be applied even when the sample data are non-independent :

$$Bias(\hat{\theta}_s) = \sum_{s=1}^{p} \kappa^{si} \sum_{j=1}^{p}\sum_{l=1}^{p}[\kappa_{ij}^{(l)} - \frac{1}{2}\kappa_{ijl}] + O(n^{-2}); \;\; s = 1, 2, \ldots, p.$$

Define $a_{ij}^{(l)} = \kappa_{ij}^{(l)} - \frac{1}{2}\kappa_{ijl}$ $\forall i, j, l = 1, 2, \ldots, p$ and define the matrices $A^{(l)} = \{a_{ij}^{(l)}\}$ $\forall i, j, l = 1, 2, \ldots, p$. After concatenating the matrices, $A = [A^{(1)}|A^{(2)}| \ldots |A^{(p)}]$, we are able to write the bias of $\hat{\theta}$ in the following way (Cordeiro and Klein, 1994):

$$Bias(\hat{\theta}) = K^{-1}Avec(K^{-1}) + O(n^{-2}).$$

Finally, define the bias adjusted-MLE as:

$$\tilde{\theta} = \hat{\theta} - \hat{K}^{-1}\hat{A}vec(\hat{K}^{-1}),$$

where $\hat{K} = K|_{\hat{\theta}}$ and $\hat{A} = A|_{\hat{\theta}}$. One of the advantages of this method is that these expressions can be evaluated when the likelihood equations for the problem in question do not admit a closed-form, analytic, solution. In such situations we can obtain bias-corrected MLE easily by means of conventional numerical methods, and $\tilde{\theta}$ is unbiased $O(n^{-2})$.

2.3. A preventive approach

We wish to obtain a bias-reduced MLE of m using a preventive approach. The MLE is typically found by solving the score equation:

$$\nabla l(\theta) = U(\theta) = 0,$$

where $l(\theta) = logL(\theta)$.

Firth (1993) proposed solving the following modified score function:

$$U_r^*(\theta) = U_r(\theta) + A_r(\theta),$$

where A may take the form $A^{(E)} = -i(\theta)b_1(\theta)/n$ or $A^{(O)} = -I(\theta)b_1(\theta)/n$. The E (O) superscript denotes expected (observed) information. In the case of the Nakagami distribution, the Hessian involves the data so we must use $A^{(E)}$.

3. Bias-reduced MLE For the Nakagami Distribution

This paper considers bias-reduction for the MLE for the μ parameter of the Nakagami distribution. Assuming independent sampling, the log-likelihood function for this problem is given as:

$$l = nln(2) + n\mu ln(\mu) - nln\Gamma(\mu) - n\mu ln(\omega) + (2\mu - 1)\sum_{i=1}^{n} ln(x_i) - \frac{\mu}{\omega}\sum_{i=1}^{n} x_i^2.$$

To proceed, we require the derivatives of the log-likelihood function up to the third order. We will also use the digamma function, defined as $\Psi(\mu) = dlog\Gamma(\mu)/d\mu$, as well as the trigamma and tetragamma functions, which are given by $\Psi_{(i)}(\mu) = d^i\Psi(\mu)/d\mu^i$ for $i = 1, 2$ respectively.

$$\frac{\partial l}{\partial \mu} = n[1 + ln(\mu)] - n\Psi(\mu) - nln(\omega) + 2\sum_{i=1}^{n} ln(x_i) - \frac{1}{\omega}\sum_{i=1}^{n} x_i^2$$

$$\frac{\partial l}{\partial \omega} = -\frac{n\mu}{\omega} + \frac{\mu}{\omega^2}\sum_{i=1}^{n} x_i^2$$

$$\frac{\partial^2 l}{\partial \mu^2} = \frac{n}{\mu} - n\Psi_1(\mu)$$

$$\frac{\partial^2 l}{\partial \mu \partial \omega} = -\frac{n}{\omega} + \frac{1}{\omega^2}\sum_{i=1}^{n} x_i^2$$

$$\frac{\partial^2 l}{\partial \omega^2} = \frac{n\mu}{\omega^2} - \frac{2\mu}{\omega^3}\sum_{i=1}^{n} x_i^2$$

$$
\begin{aligned}
\frac{\partial^3 l}{\partial \mu^3} &= -\frac{n}{\mu^2} - n\Psi_2(\mu) \\
\frac{\partial^3 l}{\partial \mu^2 \partial \omega} &= 0 \\
\frac{\partial^3 l}{\partial \omega^2 \partial \mu} &= \frac{n}{\omega^2} - \frac{2}{\omega^3}\sum_{i=1}^{n} x_i^2 \\
\frac{\partial^3 l}{\partial \omega^3} &= -\frac{2n\mu}{\omega^3} + \frac{6\mu}{\omega^4}\sum_{i=1}^{n} x_i^2
\end{aligned}
$$

We now determine the joint cumulants of the log likelihood function. Note that $E(x_i^2) = \omega$.

$$
\begin{aligned}
k_{11} &= \frac{n}{\mu} - n\Psi_1(\mu) \\
k_{12} = k_{21} &= -\frac{n}{\omega} + \frac{1}{\omega^2}\sum_{i=1}^{n} E(x_i^2) = 0 \\
k_{22} &= \frac{n\mu}{\omega^2} - \frac{2\mu}{\omega^3}\sum_{i=1}^{n} E(x_i^2) = -\frac{n\mu}{\omega^2} \\
k_{111} &= -\frac{n}{\mu^2} - n\Psi_2(\mu) \\
k_{112} &= k_{121} = k_{211} = 0 \\
k_{122} &= k_{212} = k_{221} = \frac{n}{\omega^2} - \frac{2}{\omega^3}\sum_{i=1}^{n} E(x_i^2) = -\frac{n}{\omega^2} \\
k_{222} &= -\frac{2n\mu}{\omega^3} + \frac{6\mu}{\omega^4}\sum_{i=1}^{n} E(x_i^2) = \frac{4n\mu}{\omega^3}
\end{aligned}
$$

$$
\begin{aligned}
k_{11}^{(1)} &= -\frac{n}{\mu^2} - n\Psi_2(\mu) \\
k_{11}^{(2)} &= k_{12}^{(1)} = k_{21}^{(1)} = k_{12}^{(2)} = k_{21}^{(2)} = 0 \\
k_{22}^{(1)} &= -\frac{n}{\omega^2} \\
k_{22}^{(2)} &= -\frac{2n\mu}{\omega^3}
\end{aligned}
$$

Now, to implement the Cox-Snell bias correction, we have (in the notation of section 2.1):

$$
\begin{aligned}
a_{11}^{(1)} &= -\frac{n}{2\mu^2} - \frac{1}{2} n \Psi_{(2)}(\mu) \\
a_{22}^{(1)} &= -\frac{n}{2\omega^2} \\
a_{12}^{(2)} &= a_{21}^{(2)} = \frac{n}{2\omega^2} \\
a_{11}^{(2)} &= a_{12}^{(1)} = a_{21}^{(1)} = a_{22}^{(2)} = 0
\end{aligned}
$$

$$
A^{(1)} = \begin{bmatrix} -\left(\frac{n+n\mu^2\Psi_{(2)}(\mu)}{2\mu^2}\right) & 0 \\ 0 & -\frac{n}{2\omega^2} \end{bmatrix}, \; A^{(2)} = \begin{bmatrix} 0 & \frac{n}{2\omega^2} \\ \frac{n}{2\omega^2} & 0 \end{bmatrix}.
$$

$$
K^{-1} = \begin{bmatrix} \frac{\mu}{n[\mu\Psi_1(\mu)-1]} & 0 \\ 0 & \frac{\omega^2}{n\mu} \end{bmatrix}
$$

Finally, we can write the bias vector using the result of Cordeiro and Klein (1994):

$$
Bias \begin{pmatrix} \hat{\mu} \\ \hat{\omega} \end{pmatrix} = K^{-1} A Vec(K^{-1}) + O(n^{-2}) = \begin{pmatrix} \varsigma \\ 0 \end{pmatrix} + O(n^{-2}),
$$

where $\hat{K} = K|_{\hat{\theta}}$ and $\hat{A} = A|_{\hat{\theta}}$. $\hat{\omega}$ is unbiased and $\hat{\mu}$ has $O(n^{-1})$ bias of:

$$
\varsigma = \frac{\mu\psi_{(1)}(\mu) - \mu^2\psi_{(2)}(\mu) - 2}{2n(\mu\psi_{(1)}(\mu) - 1)^2}.
$$

It is readily verified that the bias is positive for all $\mu > 0$. Thus, the bias-reduced MLE of μ is given as:

$$
\tilde{\mu} = \hat{\mu} - \frac{\hat{\mu}\psi_{(1)}(\mu) - \hat{\mu}^2\psi_{(2)}(\hat{\mu}) - 2}{2n(\hat{\mu}\psi_{(1)}(\hat{\mu}) - 1)^2},
$$

which is unbiased to $O(n^{-2})$.

To implement Firth's (1993) "preventive" adjustment of the score function, we note that, in our problem, $\theta = \{(r_1, r_2) : \mu, \omega \in R_+\}$. However we know that the MLE for ω, namely $\hat{\omega} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$, is unbiased. Noting that

$$
A^{(E)} = -i(\theta) b_1(\theta)/n = -i(\theta) K^{-1} A vec(K^{-1}) = -A vec(K^{-1}),
$$

the modified score function to solve (for $\hat{\mu}$) is:

$$U_r(\theta) = -A_r(\theta) = Avec(K^{-1}),$$

or

$$nln(\hat{\mu}) - n\psi(\hat{\mu}) - nln\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) + \sum_{i=1}^{n} ln(x_i^2) = -\frac{1+\hat{\mu}^2\psi_{(2)}(\hat{\mu})}{2(\hat{\mu}^2\psi_{(1)}(\hat{\mu})-\hat{\mu})} + \frac{1}{2\hat{\mu}}.$$

This non-linear function of $\hat{\mu}$ can be solved easily by standard numerical methods.

4. Numerical evaluations

The corrective and preventive analytic bias reduction methods of Cox and Snell and Firth, respectively, eliminate the bias to $O(n^{-1})$. The resulting estimators are still biased to $O(n^{-2})$. We have conducted a Monte Carlo simulation experiment to examine the bias and mean squared error (MSE) of the basic MLE for the shape parameter, μ, of the Nakagami distribution, as well as these measures for the corrective and preventive MLEs, $\tilde{\mu}$ and μ^f. An alternative, computationally intensive, way of dealing with the bias of the MLE to the same order of magnitude is to use the parametric bootstrap to simulate the bias, and then bias-adjust the MLE. This approach is also corrective in nature, and we compare its performance here with the effectiveness of the two analytic bias-reduced MLEs. The bootstrap bias-corrected estimator is obtained as $\breve{\mu} = 2\hat{\mu} - (1/N_B)\sum_{j=1}^{N_B}\hat{\mu}_{(j)}$, where $\hat{\mu}_{(j)}$ is the MLE of μ obtained from the j^{th} of the N_B bootstrap samples. See Efron (1982, p.33).

The simulations were undertaken using the R statistical software environment (R, 2008). Nakagami-distributed variates were generated using the VGAM package (Yee, 2009), and the log-likelihood function was maximized using the nleqslv package (Hasselman, 2009). Each part of the experiment uses 50,000 Monte Carlo replications. In the case of $\breve{\mu}$ we use 1,000 bootstrap samples per replication (i.e., 50 million evaluations for each value of n, in this case). The results that are reported in Table 1 are percentage biases and MSEs, the latter being defined as $100 \times (MSE/\mu^2)$. For each of the estimators under consideration the percentage biases and MSEs are invariant to the value of the scale parameter, ω, for a given sample size, so we assign $\omega = 1$.

A range of values of $\mu \geq 0.75$ is considered, including $\mu = 1$, which corresponds to the Rayleigh distribution. When smaller values of μ were included in the experiment it was found that a substantial number of the Monte Carlo replications resulted in estimates for the shape parameter which were less than the admissible lower bound of 0.5. This problem was, not surprisingly, compounded significantly in the case of the bootstrap-corrected estimator. We comment further on this below, when offering our recommendations about the choice of bias-reduction technique to adopt in practice.

The simulation results appear in Table 1. We see that the relative bias of the MLE, $\hat{\mu}$, can be quite substantial for small sample sizes, especially in the "post-Rayleigh" range for μ. The relative bias and MSE increase with the value of μ, and of course they decrease with n, as the estimator is consistent. Both of the analytic techniques for handling the bias of $\hat{\mu}$ are equally successful, and they reduce the percentage bias by one or two orders of magnitude, virtually eliminating it in most of the cases considered. These two estimators, $\tilde{\mu}$ and μ^f, also have slightly lower percentage MSE than the original estimator, $\hat{\mu}$. Overall, there is nothing to choose between the corrective and preventive analytic bias reduction methods for this problem.

The bootstrap bias correction also succeeds in reducing the bias of $\hat{\mu}$, but it over-corrects when $n = 25$. For small sample sizes, the remaining relative (absolute) bias for $\breve{\mu}$ is an order of magnitude greater than that for $\tilde{\mu}$ and μ^f. For $n \geq 100$, $\breve{\mu}$ often out-performs $\tilde{\mu}$ and μ^f in terms of absolute percentage bias (*e.g.*, when $\mu = 15$ and $n = 200$). However, in those cases where the bootstrap-based estimator might be preferred, all three bias-corrected estimators have negligible percentage bias. The relative MSE for $\breve{\mu}$ is always very similar in magnitude to that of both $\tilde{\mu}$ and μ^f. In summary, we recommend that either the Cox-Snell "corrective" estimator, or Firth's "preventive" estimator be used for the Nakagami shape parameter, given their performance with respect to both bias and MSE, and their computational simplicity.

Table 1. Percentage biases and MSE's of shape parameter estimators

n	$\%Bias(\hat{\mu})$ $\%MSE(\hat{\mu})$	$\%Bias(\tilde{\mu})$ $\%MSE(\tilde{\mu})$	$\%Bias(\mu^f)$ $\%MSE(\mu^f)$	$\%Bias(\breve{\mu})$ $\%MSE(\breve{\mu})$
		$\mu = 0.75$		
25	10.842	0.207	0.295	-1.867
	[10.304]	[7.180]	[7.189]	[7.764]
50	5.036	0.056	0.076	-0.343
	[3.885]	[3.237]	[3.238]	[3.320]
100	2.466	0.050	0.055	-0.018
	[1.684]	[1.534]	[1.535]	[1.535]
200	1.252	0.062	0.064	0.045
	[0.790]	[0.753]	[0.752]	[0.753]
		$\mu = 1.0$		
25	11.071	-0.088	-0.009	-1.585
	[10.812]	[7.492]	[7.498]	[7.277]
50	5.090	-0.137	-0.118	-0.465
	[4.098]	[3.410]	[3.410]	[3.388]
100	2.590	0.050	0.055	-0.024
	[1.784]	[1.620]	[1.620]	[1.620]
200	1.279	0.029	0.029	0.011
	[0.836]	[0.796]	[0.796]	[0.797]
		$\mu = 2.0$		
25	12.235	-0.015	0.033	-1.678
	[12.558]	[8.584]	[8.585]	[8.307]
50	5.764	0.026	0.038	-0.335
	[4.618]	[3.791]	[3.791]	[3.768]
100	2.797	0.017	0.019	-0.069
	[2.003]	[1.812]	[1.812]	[1.811]
200	1.340	-0.029	-0.028	-0.049
	[0.935]	[0.890]	[0.890]	[0.890]

Table 1. Percentage biases and MSE's of shape parameter estimators (continued)

n	$\%Bias(\hat{\mu})$ $\%MSE(\hat{\mu})$	$\%Bias(\tilde{\mu})$ $\%MSE(\tilde{\mu})$	$\%Bias(\mu^f)$ $\%MSE(\mu^f)$	$\%Bias(\breve{\mu})$ $\%MSE(\breve{\mu})$
		$\mu = 5.0$		
25	12.947	-0.091	-0.072	-1.872
	[13.483]	[9.1460]	[9.146]	[8.866]
50	6.274	0.156	0.160	-0.230
	[5.202]	[4.250]	[4.250]	[4.222]
100	3.007	0.046	0.047	-0.045
	[2.210]	[1.994]	[1.994]	[1.992]
200	1.347	-0.109	-0.108	-0.129
	[1.009]	[0.962]	[0.962]	[0.963]
		$\mu = 10.0$		
25	13.251	-0.076	-0.067	-1.888
	[14.064]	[9.532]	[9.532]	[9.241]
50	6.292	0.046	0.0479	-0.349
	[5.339]	[4.368]	[4.368]	[4.340]
100	3.097	0.069	0.070	-0.023
	[2.267]	[2.043]	[2.043]	[2.042]
200	1.494	0.005	0.005	-0.015
	[1.037]	[0.984]	[0.984]	[0.985]
		$\mu = 15.0$		
25	13.273	-0.144	-0.138	-1.957
	[14.117]	[9.568]	[9.568]	[9.311]
50	6.345	0.053	0.054	-0.352
	[5.342]	[4.365]	[4.365]	[4.335]
100	2.939	-0.105	-0.105	-0.120
	[2.265]	[2.050]	[2.050]	[2.049]
200	1.539	0.038	0.038	0.017
	[1.053]	[0.999]	[0.999]	[1.000]

Some additional comments of a practical nature are also in order. As the MLE for the shape parameter is upward-biased, correcting by using either $\tilde{\mu}$ and μ^f will always reduce the value of the point estimate. If the original MLE is close to 0.5 in value, this may result in values for $\tilde{\mu}$ and/or μ^f which violate the constraint $\hat{\mu} \geq 0.5$. For example, when μ was chosen to be 0.5001 in a Monte Carlo experiment, we found that 42% of the time $\hat{\mu} \leq 0.5$ when $n = 25$. This percentage approached 50% as the sample size increased (49% at $n = 5000$), reflecting the asymptotic normality and consistency of the MLE. In such instances one might either reject the Nakagami model, or one might impose the boundary value as the point estimate. Our preference is to follow the second of these options if this situation arises in a practical application.

5. Empirical examples

5.1 Seismological data

Our first example relates to a recent application of the Nakagami distribution to seismological data by Nakahara and Carcolé (2010). Using data relating to 60 earthquakes from three recording sites in the Katakami Mountain Range in Eastern Japan, these authors estimate the shape parameter of the Nakagami distribution, using MLE, and use this information to estimate seisomogram envelopes. Previous analyses of this type use the Rayleigh distribution, and Nakahara and Carcolé find that generalizing this to the Nakagami distribution has important implications for the results.

Using the MLEs and asymptotic standard errors (a.s.e.) in Table 1 of Nakahara and Carcolé (2010), and their equation (19), we are able to impute the sample sizes used for the various results that are tabulated. We have bias-adjusted the MLE for the shape parameters for the results at the $1-2$ Hz. frequency, using the Cox-Snell corrective procedure. Acceleration measurements at the three recording stations are taken at the top and the bottom (with the character 'B' in the site name) of a 100m borehole; and in the vertical, North-South, and East-West directions. So, there are results for 18 cases reported in Table 2 below. In all cases, the original MLEs are in the post-Rayleigh range, but only three of our bias-adjusted estimates are in this range. However, the numerical differences are not statistically significant when the bootstrap standard errors (b.s.e.) for the latter estimates are taken into account. The p-values reported in the last column of Table 2 relate to the occurrence of values, in the sampling distribution of $\tilde{\mu}$ that are as large, or larger, than the corresponding point estimate based on $\hat{\mu}$. These values also imply that there is no significant difference between bias-adjusted and unadjusted estimates of the shape parameter.

Table 2: Fitting the Nakagami distribution to seismic recordings (1 − 2Hz.)

Site	n	$\hat{\mu}$	a.s.e.	$\tilde{\mu}$	b.s.e.	p
IWTH13						
East-West	18	1.04	0.31	0.90	0.36	0.41
North-South	18	1.12	0.33	0.96	0.38	0.41
Vertical	16	1.09	0.34	0.92	0.42	0.40
IWTH17						
East-West	53	1.05	0.18	1.00	0.19	0.46
North-South	19	1.08	0.31	0.94	0.36	0.42
Vertical	67	1.05	0.16	1.01	0.17	0.45
IWTH02						
East-West	67	1.05	0.16	1.01	0.17	0.45
North-South	11	1.18	0.45	0.91	0.58	0.38
Vertical	22	1.11	0.30	0.98	0.34	0.42
IWTH13						
East-West	11	1.07	0.41	0.83	0.56	0.37
North-South	16	1.12	0.35	0.95	0.43	0.41
Vertical	12	1.13	0.41	0.89	0.54	0.39
IWTH17						
East-West	19	1.09	0.31	0.95	0.37	0.41
North-South	60	1.05	0.17	1.01	0.18	0.46
Vertical	84	1.03	0.14	1.00	0.14	0.46
IWTH02						
East-West	8	1.23	0.54	0.84	0.94	0.34
North-South	9	1.26	0.53	0.90	0.75	0.36
Vertical	9	1.24	0.52	0.89	0.80	0.36

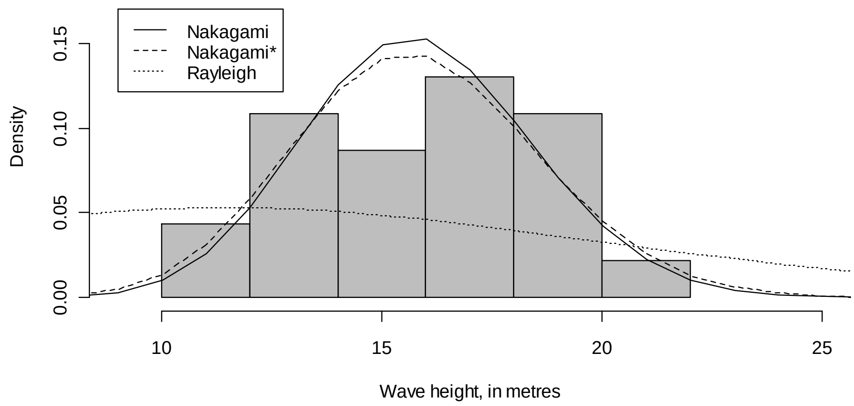
Note: Sample sizes are imputed from Table 1 and equation (19) of Nakahara and Carcolé (2010).

5.2. Wave height data

The modeling of maximum sea wave heights is of considerable importance in ocean engineering, and other related fields. For example, such models are used to provide input into the construction specifications for fixed-location offshore structures such as oil and gas rigs, as well as for ocean-going vessels. The underlying physics of random waves on a Gaussian free surface elevation suggests that (appropriately measured) wave heights will follow a Rayleigh distribution (Longuet-Higgins, 1952). That is, they should follow a Nakagami distribution with $\mu = 1$. However, in practice ocean waves are observed to be asymmetric as a result of underlying nonlinearities in their generating mechanism. The Rayleigh distribution is very restrictive, having a single (scale) parameter. Forristall (1978) and Magnusson *et al.* (1999) note that the Rayleigh distribution tends to over-predict the heights of the largest waves. For a fixed value of the scale parameter, the right-tail probability of the Nakagami distribution declines as μ increases. This suggests that a Nakagami distribution with $\mu > 1$, and a second (scale) parameter, may out-perform the Rayleigh distribution when modeling the heights of rogue waves.

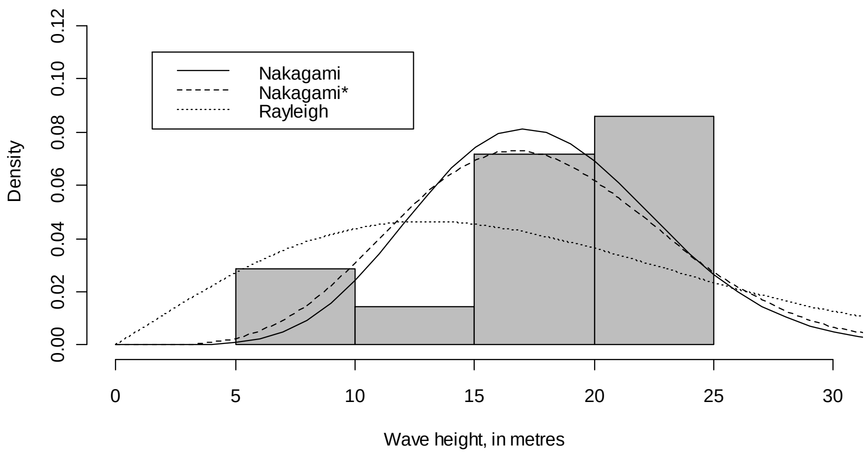
We present two applications of the Nakagami distribution to this problem. We believe that this is the first time that this distribution has been used in this particular context, and work in progress investigates its usefulness in more detail. The first set of data that we use are for the maximum down-crossing wave heights ($H_{max\,D}$), in metres, for 23 abnormal waves, as reported by Petrova *et al.* (2006, p. 237). These waves were measured at the offshore platform North Alwyn in the northern part of the North Sea, about 100 miles east of the Shetland Islands, during the November storm in 1997. The second set of data are for the zero down-crossing wave heights (peak plus trough), H^*, in metres, for 14 freak waves measured at the same North Sea location, as reported by Stansell (2005, p. 1018). Figures 1 and 2 show empirical and fitted distributions for the Petrova *et al.* and Stansell data-sets, respectively. Given the small sample sizes, we anticipate that correction for bias in the estimation of the shape parameter of the Nakagami distribution will be important.

Figure 1: Petrova *et al.* (2006) data, empirical and fitted distributions



Note: Nakagami (Nakagami*) refers to the distribution fitted with the MLE (corrective bias-adjusted MLE).

Figure 2: Stansell (2005) data, empirical and fitted distributions



Note: Nakagami (Nakagami*) refers to the distribution fitted with the MLE (corrective bias-adjusted MLE).

Table 3 reports the maximum likelihood estimates of the scale and shape parameters, together with the Cox-Snell and Firth modified estimators of the latter parameter, for these two data-sets. Asymptotic standard errors are provided for $\hat{\omega}$ and $\hat{\mu}$. For a generic parameter, θ, asymptotic standard errors are given by the square-roots of the diagonal elements of the inverse information matrix, $I^{-1}(\hat{\theta})$, where the latter is defined as $-\left\{E\left[\left(\partial^2 logL(\theta)\right)/\left(\partial\theta\partial\theta'\right)\right]\right\}|_{\theta=\hat{\theta}}$. We also report 95% confidence intervals for the parameters. These are bootstrap percentile intervals, based on 999 bootstrap samples. The latter number is justified by the results of Efron (1987, p.181). The results support the use of the Nakagami distribution, as the estimates of the shape parameter are significantly different from the (Raleigh) value of unity. In addition, the reductions in the values of the shape parameter estimates, induced by the bias corrections, are numerically sizeable. However, we see from the bootstrap confidence intervals that these reductions are not statistically significant here.

Table 3. Fitting the Nakagami distribution to wave data

	$\hat{\omega}$	$\hat{\mu}$	$\tilde{\mu}$	μ^f
		Petrova *et al.* data ($n = 23$)		
estimate	258.527	9.499	8.289	8.290
a.s.e.	(17.490)	(2.753)	-	-
b.s.e.		(3.470)	(3.184)	(2.739)
95% b.c.i.		[6.025, 19.083]	[5.293, 17.777]	[4.487, 15.347]
		Stansell data ($n = 14$)		
estimate	341.643	3.441	2.749	2.753
a.s.e.	(49.222)	(1.242)	-	-
b.s.e.		(1.890)	(1.394)	(1.195)
95% b.c.i.		[2.061, 9.480]	[1.591, 7.036]	[1.326, 5.763]

Note: 95% confidence intervals appear in brackets. These are bootstrap percentile intervals, to allow for the non-normality of the small-sample distributions of the MLEs.

6. Conclusions

The Nakagami distribution is widely used in a number of disciplines, especially in the analysis of the fading of radio and ultrasound signals. Recently, it has also been finding application in other fields, including hydrology, seismology, and data compression. Accurate estimation of the shape parameter of this distribution is crucial in all of these applications. For example, this parameter is a direct measure of the fading rate in the case of radio waves; and in seismology its reciprocal is the scintillation index, which is used to express fluctuations in seismogram envelopes.

While it is well known that the maximum likelihood estimator of the scale parameter in the Nakagami distribution is unbiased, we find that the corresponding estimator is upward-biased in finite samples. In addition, we have shown that this bias can be dramatically reduced by adjusting the estimator by analytic methods of either a "preventive" or "corrective" nature. These techniques are simple to apply, and are equally successful in constructing estimators that are unbiased to $O(n^{-2})$, where n is the sample size. Our simulation results show that a parametric bootstrap bias correction is also very effective, and that in all three cases the reduction in bias is accompanied by a small improvement in mean squared error. When computational costs are taken into account, we recommend the use of either the Cox-Snell (1968) "corrective" procedure or the Firth (1993) "preventive" procedure, to obtain an almost-unbiased maximum likelihood estimator for the shape parameter of the Nakagami distribution.

Finally, we have illustrated the practical effect that undertaking this bias adjustment can have in two empirical applications. In the case of some seismic acceleration data we find that the shape parameter estimates are largely modified from the post-Rayleigh ($\mu > 1$) range to the pre-Rayleigh ($\mu < 1$) range as a result of using the Cox-Snell bias adjustment. Our other example provides a novel application of the Nakagami distribution to ocean wave height data. We find that the results support the use of this distribution (rather than the conventional Rayleigh distribution) in this context, and both of the analytic bias-correction procedures result in marked numerical reductions in the estimates of the shape parameter. Work in progress pursues our investigation of the usefulness of the Nakagami distribution for modeling the characteristics of ocean waves.

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