

Department of Economics

***IDENTIFYING THE CYCLE OF A MACROECONOMIC TIME-SERIES  
USING FUZZY FILTERING***

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**December, 2004**

**Abstract**

This paper presents a new method for extracting the cycle from an economic time series. This method uses the fuzzy c-means clustering algorithm, drawn from the pattern recognition literature, to identify groups of observations. The time series is modeled over each of these sub-samples, and the results are combined using the “degrees of membership” for each data-point with each cluster. The result is a totally flexible model that readily captures complex non-linearities in the data. This type of “fuzzy regression” analysis has been shown by Giles and Draeseke (2003) to be highly effective in a broad range of situations with economic data. The fuzzy filter that we develop here is compared with the well-known Hodrick-Prescott (HP) filter in a Monte Carlo experiment, and the new filter is found to perform as well as, or better than, the HP filter. The advantage of the fuzzy filter is especially pronounced when the data have a deterministic, rather than stochastic, trend. Applications with real time-series illustrate the different conclusions that can emerge when the fuzzy regression filter and the HP filter are each applied to extract the cycle.

**Keywords:** HP filter, fuzzy clustering, business cycle, trend extraction

**JEL Classifications:** C19; C22; E32;

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## 1. Introduction

Extracting the trend or cycle from an economic time-series is an important problem that arises in many contexts. Examples arise in the literature deals with testing for asymmetries in the business cycle (*e.g.*, Neftçi, 1984; Sichel, 1989, 1993; Harvey and Jaeger, 1993, Giles, 1997; Verbrugge, 1997; Speight and McMillan, 1998; Canova, 1998; Razzak, 2001; Corbae and Ouliaris, 2002; and Psaradakis and Sola, 2003); and in the context of measuring potential output (*e.g.*, Giorno *et al.*, 1995; Scott, 2000; Grounder and Morling, 2000; Orphanides and van Norden, 2001; Haltmaier, 2001; van Norden, 2002; Rennison, 2003; and Ögünç and Ece, 2004). One of associated issues is that many macroeconomic time-series are non-stationary, and this needs to be taken into account when a filter is chosen to extract the cyclical component.

Various filters have been proposed and widely used in this context, including the Beveridge and Nelson (1981) filter (hereafter BN), the Baxter and King (1999) filter (BK), and the Hodrick and Prescott (1980, 1997) filter (HP). The latter is perhaps the most widely used, and appropriately it has been subjected to considerable scrutiny. It has not escaped this scrutiny unscathed, and some of its limitations are well-documented.

In this paper we investigate the performance of a new approach to handling this filtering problem – one based on “fuzzy regression” analysis. This general semi-parametric approach has been outlined in detail recently by Giles and Draeseke (2003). They illustrate its application in a variety of different situations, and they point out that it can be used for trend extraction. Their methodology has several potential merits. It allows for very flexible trends that may be quite non-linear. No functional form is assumed in advance. Rather, the characteristics and patterns of the data determine this. Fuzzy regression is very easy to apply, especially in the context under consideration here, and our purpose is to evaluate this technique by means of a Monte Carlo experiment, and to compare its performance with that of the HP filter.

In section 2 we briefly describe the HP filter and note some of its features and shortcomings. Section 3 describes fuzzy regression and the way it is to be applied to our filtering problem; and in section 4 the details of the Monte Carlo experiment are provided. The results of this experiment are discussed in section 5, and some illustrative applications are presented in section 6. Section 7 concludes the paper.

## 2. The HP Filter

The purpose of the HP filter is to decompose a time-series,  $\{y_t\}$  into two additive components – a cycle,  $\{y_t^c\}$ , and a growth component  $\{y_t^g\}$ . If the time-series is measured monthly or quarterly, it is assumed that any deterministic seasonality has been removed already, although knowledge of the data frequency will be of some importance. Usually, with macroeconomic data, a multiplicative decomposition will be appropriate, so  $\{y_t\}$  will measure the logarithm of the original data. The basic rationale behind the HP filter is that we make the decomposition so as to minimize the variation of the cyclical component, subject to a penalty for the amount of variation that is left in the second difference of the growth component. Other interpretations of the motivation behind the HP filter can also be found in the literature.

So,  $\{y_t^g\}$  is obtained as:

$$\{y_t^g\}_{t=0}^{T+1} = \arg \min \sum_{t=1}^T \{ (y_t - y_t^g)^2 + \lambda [(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)]^2 \} \quad (1)$$

The larger is the smoothness parameter,  $\lambda$ , the smoother is the growth component, and in the limit this component becomes a linear time trend. Choosing a value for the smoothness parameter is an important issue when applying the HP filter, and this topic has attracted a sizeable literature. Hodrick and Prescott (1980, 1997) suggest a value of 1600 in the case of quarterly time-series data. Other choices are suggested by various authors (*e.g.*, Backus and Kehoe, 1992; Correia *et al.*, 1992, and Ravn and Uhlig, 1997) in the cases of monthly or annual data. More recently, Schlicht (2004) has suggested a procedure for estimating this parameter from the data themselves.

Although it has been widely used, there are a number of unsatisfactory properties of the HP filter that need to be taken into consideration. Apart from the dependence of the results on the value that is chosen for the smoothing parameter, the filter can also give rise to distorted results at the end-points of the sample. This is because it is a two-sided filter, as can be seen in equation (1). Accordingly, some authors (*e.g.*, Baxter and King, 1995) suggest that the first and last three years of HP-filtered data should be discarded, and recently Bruchez (2003) has proposed a modification to the filter to alleviate this problem. Of course, such distortions are unfortunate as the most recent part of the time-series is also the most interesting or important in some applications. Moreover, Cogley and Nason (1995) show that the HP filter can generate cyclical dynamics even when none are really present in the original data. That is to say, they can “create” business cycles that do not really exist, a point that we return to in discussing our own empirical applications in

section 6 below. On a more positive note, King and Rebelo (1993) show that if the HP filter is applied to non-stationary data that have an order of integration of no more than four, the filtered data will be stationary. This is especially important when working with macroeconomic data. The HP filter is coded into most of the standard econometrics packages, and in our study we use the SHAZAM (2001) package for all of our computations.

### 3. Fuzzy Regression and the Fuzzy Filter

Fuzzy regression, as outlined by Giles and Draeseke (2003) involves three stages. First, the sample data are partitioned into a number of “fuzzy clusters”. Second, a regression model is fitted (using any appropriate estimation technique, such as least squares, instrumental variables, non-linear maximum likelihood) separately over each cluster. Finally, the fitted models for each cluster are combined in a weighted average fashion, the weights being the “degrees of membership” that associate each of the original sample points with each of the fuzzy clusters. Because these weights vary continuously through the sample, even if linear models are fitted to each cluster, the weighted average model can capture very complex non-linearities with ease. Essentially, it is an application of the Takagi and Sugeno (1985) approach to fuzzy systems, as is noted by Shepherd and Shi (1998).

We use the well-known “fuzzy c-means” (FCM) algorithm to partition the “ $n$ ” data-points into “ $c$ ” fuzzy clusters (where  $c < n$ ), while simultaneously determining the locations of these clusters in the appropriate space.<sup>1</sup> The data may be multi-dimensional, and the metric that forms the basis for the usual FCM is “squared error distance”. Let  $x_k$  be the  $k$ 'th. (possibly vector) data-point ( $k = 1, 2, \dots, n$ ). Let  $v_i$  be the center of the  $i$ 'th. (fuzzy) cluster ( $i = 1, 2, \dots, c$ ). Let  $d_{ik} = \|x_k - v_i\|$  be the distance between  $x_k$  and  $v_i$ ; and let  $u_{ik}$  be the “degree of membership” of data-point “ $k$ ” in cluster “ $i$ ”, where :

$$\sum_{i=1}^c (u_{ik}) = 1$$

The objective is partition the data-points into the “ $c$ ” clusters, and simultaneously locate those clusters and determine the associated “degrees of membership”, so as to minimize the functional

$$J(u, v) = \sum_{i=1}^c \sum_{k=1}^n (u_{ik})^m (d_{ik})^2.$$

There is no prescribed manner for choosing the exponent parameter, “ $m$ ”, which must satisfy  $1 < m < \infty$ . In practice,  $m = 2$  is a common choice, and the one that we adopt here. Noting the

relationship between the  $d_{ik}$ 's and the  $v_i$ 's above, the membership values and the centres of the fuzzy clusters are obtained iteratively as:

$$u_{ik} = 1 / \left\{ \sum_{j=1}^c [(d_{ik})^2 / (d_{jk})^2]^{1/(m-1)} \right\}.$$

$$v_i = \left[ \sum_{k=1}^n (u_{ik})^m x_k \right] / \left[ \sum_{k=1}^n (u_{ik})^m \right] ; i = 1, 2, \dots, c$$

The algorithm is readily programmed and the fixed-point nature of this problem ensures the existence of a solution.

The fuzzy regression model is of the form:

$$y = f(x) + \varepsilon$$

where the functional relationship will typically involve unknown parameters, and  $\varepsilon$  is a random disturbance term.<sup>3</sup> The identification and estimation of the fuzzy model then proceeds according to the following steps:

- Partition the sample observations for  $x$  into  $c$  fuzzy clusters, using the FCM algorithm.
- Fit the model over each fuzzy cluster separately:

$$y_{ij} = f_i(x_{ij}) + \varepsilon_{ij} ; j = 1, \dots, n_i ; i = 1, \dots, c$$

In particular, if the chosen estimation procedure is linear least squares, then

$$y_{ij} = \beta_{i0} + \beta_{i1}x_{ij} + \varepsilon_{ij} ; j = 1, \dots, n_i ; i = 1, \dots, c$$

- Model and predict the conditional mean of  $y$  using:

$$\hat{y}_k = \left[ \sum_{i=1}^c (\hat{\beta}_{i0} + \hat{\beta}_{i1}x_k) u_{ik} \right] / \left[ \sum_{i=1}^c u_{ik} \right] ; k = 1, \dots, n$$

where a “ $\hat{\phantom{x}}$ ” denotes a least squares estimator.

So, the fuzzy predictor of the conditional mean of  $y$  is a weighted average of the linear predictors based on the fuzzy partitioning of the explanatory data, with the weights (membership values) varying continuously through the sample. This latter feature enables non-linearities to be modelled effectively. Giles and Draeseke (2003) illustrate the application of the above methodology to a number of econometric modeling problems.<sup>4</sup> Among other things, they also note that if the explanatory variable is chosen to be a linear time trend, then a fuzzy regression analysis that models  $y_t$  as a function of this  $x_t$  is one way of “de-trending” the series. This provides a competitor for the HP filter, and the other techniques that have been suggested, such as

the BK and BN filters. This paper evaluates the relative performances of this fuzzy regression (FR) filter and the HP filter in a Monte Carlo experiment.

#### 4. Monte Carlo experiment design

Our experiment involved the following data-generating process:

$$\begin{aligned}
 y_t &= y_t^s + y_t^c \\
 y_t^s &= \alpha y_{t-1}^s + \varepsilon_t \\
 y_t^c &= \theta_1 y_{t-1}^c + \theta_2 y_{t-2}^c + \eta_t \\
 \varepsilon_t &\sim i.i.d. N(0, \sigma_\varepsilon^2) \quad ; \quad \eta_t \sim i.i.d. N(0, \sigma_\eta^2) \\
 \varepsilon_t, \eta_t &\text{ independent, } \forall t, \tau.
 \end{aligned}$$

When  $\alpha = 1$  we have a non-stationary stochastic trend, and this is the situation considered by Guay and St.-Amant (1997) in their study of the HP and BK filters. This is then the data-generating process suggested by Watson (1986) as a representation of real U.S. GNP. Because the cyclical component of the time-series is represented by a second-order autoregressive process, the peak of the spectrum for the series could be at the zero frequency or at the usual business-cycle frequencies. Within this framework we also consider a deterministic trend, by setting  $|\alpha| < 1$ .

We have examined various values for the moving average parameters for the cycle, and for the ratio of the error variances. We set  $T = 100$ ; and used 5,000 replications in the Monte Carlo experiment.<sup>5</sup> The FR filter was based on  $c = 2$  clusters, and the smoothing parameter for the HP filter was set to 1600. The first and last eight filtered data-points were discarded in recognition of the likely poor performance of the HP filter at the ends of the sample.

#### 5. Monte Carlo results

Tables 1 to 3 present our results in the case of the non-stationary stochastic trend. The performances of the HP and FR filters are assessed in various ways: (i) through the correlation between the extracted cyclical component and the true cycle; (ii) by comparing the standard deviation of each extracted cyclical component with the standard deviation of the true cycle over the sample period; and (iii) by making similar comparisons between the autocorrelations and the partial autocorrelations.

Both the HP and FR filters perform poorly in the top portion of Table 1, which corresponds to series with a strong permanent component. In the regions of Table 1 where the correlations between the true and filtered cycles are statistically significant, we see that the FR filter performs increasingly better than the HP filter when the noise in the cyclical component noise comes to dominate that of the permanent component. When there is a relatively strong permanent component in the time-series, both the HP and FR filters perform very poorly when the estimated and true standard deviations of the cycle are compared. Unless the noise ratio is very small (in the lower part of Table 1), the HP filter generally out-performs the FR filter on the basis of this measure.

Table 2 shows the autocorrelation functions for the cycles extracted using both the HP and FR filters. Again, 95% confidence intervals are given in parentheses below each result, and in this case the true values of the autocorrelations appear in square brackets alongside the point estimates. We see that there are negligible differences between the simulated values of the autocorrelation functions when the noise ratio is equal to 10. As was the case in Table 1, when the noise ratio is equal to 0.01, the FR filter performs better, regardless of the values used to define the cyclical component of the series. Table 3 reports corresponding information relating to the partial autocorrelation functions for both the two filters. Here, there is no clear pattern in terms of which method performs better. However, the FR filter performs marginally better than the HP filter in the bottom part of the table, which relates to series with a dominant cyclical component.

Tables 4 to 6 provide corresponding results when the time-series has a stationary deterministic trend. The layout of these tables is the same as for the non-stationary case. These results are much more supportive of the FR filter, and indicate that in this case it has some important advantages over the HP filter. In Table 4, the correlations between the actual cycle and the FR extracted cycle are always greater than those associated with the HP filter. For both filters, the cycle standard deviation ratios are closer to unity than was the case in Table 1, and in general the FR filter dominates. Moreover, in those cases where the ratio associated with the HP filter is closer to unity than that associated with the FR filter, there is very little difference between them. The results in Tables 5 and 6 show that the FR filter invariably captures the autocorrelation and partial autocorrelation functions of the true cycle more accurately than does the HP filter in this stationary case.

Neither of the two filters perform well in situations where there is an important permanent component in the time-series. Guay and St. Amant (1997) demonstrate that the HP and BK filters are able to characterize situations where the peak of the spectrum for  $\{y_t\}$  is located at business cycle frequencies, but they fail badly when this peak is at the zero frequency. The latter situation is characteristic of most non-stationary macroeconomic time-series, and corresponds to Granger's (1966) "typical spectral shape" for such data.<sup>6</sup> Our new FR filter also performs badly in such circumstances. However, the FR filter clearly outperforms the HP filter in exactly those situations where the latter is at its best – namely when the time-series in question is dominated by the cycle, rather than the trend. Moreover, our results also indicate that the new fuzzy regression filter dominates the HP filter in the case of a stationary time-series. Of course, this has important implications for its application.

## 6. Empirical applications

As a first illustration of the application of the new FR filter we consider the series for U.S. real quarterly GDP, 1952Q1 to 2004Q2, a series that is well-known to be strongly trended and non-stationary.<sup>7</sup> The latter characteristic is confirmed by the augmented Dickey-Fuller (ADF) and KPSS test results in Table 7a. We illustrate the sensitivity of the FR cycle to the numbers of fuzzy clusters that are used in its construction - Figures 1a and 1b are for  $c = 2$  and  $c = 3$  respectively.

The two FR filtered cycle series exhibit a little more amplitude than does the HP series, commensurate with the fact that the growth component identified by the HP filter is more volatile than that determined by the FR filter. The variability in the two cycles may be compared in various ways. For example, measuring variability in terms of the coefficient of variation to account for the different mean values (and the different units across the following two examples), we see from Table 7b that the HP cycle is slightly more variable than are the FR cycles.<sup>8</sup> The timing of the major turning points in the HP and FR cycles matches closely. However,, when we count *all* of the turning points, we see in Table 7b that the HP cycle has 44 peaks and 45 troughs, compared with 36 and 37 respectively for both of the cycles obtained through the FR filter. Although the turning points of the FR cycles are robust to the choice of " $c$ ", the cycle's amplitude reduces somewhat as the number of fuzzy clusters is increased. This is anticipated because the FR cycle is the difference between the original data and the fitted "fuzzy regression", and it is always the case that the latter becomes more non-linear as " $c$ " increases.<sup>9</sup> So, by either of these measures, the FR filter produces a cycle that is somewhat less variable than that generated by the HP filter.



Recalling, from section 2, that Cogley and Nason (1995) show that the HP filter can generate cyclical dynamics even when none are really present in the original data, one might conclude that at least for this series the FR filter produces a more reliable measure of the cycle.

Our second example involves the series for U.S. quarterly real personal consumption expenditure on durables, 1947Q1 to 1971Q4.<sup>10</sup> The logarithm of the series, which has a pronounced upward trend, appears in Figure 2a. This series also has a unit root, as can be seen from the values of the ADF and KPSS statistics in Table 7a, and it also has Granger's "typical spectral shape", corresponding to the upper parts of the tables of Monte Carlo results cited in section 5. The HP and FR cycles appear in Figure 2b, and their characteristics are very similar to those of their counterparts in Figure 1b. In particular, the FR cycle has somewhat more amplitude than does the HP cycle, and the timing of the major turning points match closely. In this case the coefficients of variation of the HP and FR cycles are 1.31% and 1.39% respectively, so the FR filter results in a slightly more variable cycle. There is really no difference between the number of peaks (troughs) for the two cycles. The results that we present are only for  $c = 2$ , in keeping with this choice in the Monte Carlo experiment. However, these results are insensitive to setting  $c = 3$ .

The final series that we consider is quarterly Canadian capacity utilization for the period 1962Q1 to 2001Q2.<sup>11</sup> Figure 3a depicts the logarithm of this series, which is highly cyclical and is found to be *stationary* according to both the augmented Dickey-Fuller and KPSS tests. The extent to which the results associated with these two filters differ is shown in Figure 3b. Once again we see that there is substantial agreement with respect to the major turning points of the two series. As can be seen from Table 7b, the HP cycle is more variable than is the FR cycle, in terms of both the coefficient of variation and the total number of turning points in the series. On the basis of the Monte Carlo evidence cited above, we might anticipate that the FR filter may be more appropriate than the HP filter for extracting the cycle of this series, given that it is stationary. This point is reinforced if we consider the variability measures in the context of the Cogley and Nason (1995) findings.

## 7. Conclusions

In this paper we have proposed and investigated a new filter that can be used to extract the cyclical component from an economic time-series. This filter is based on the "fuzzy regression" methods developed by Giles and Draeseke (2003), and is very simple to apply as it involves only

one-dimensional fuzzy clustering and simple least squares regression analysis. The Monte Carlo evidence that we have presented suggests that the performance of this new filter is somewhat mixed. Although it performs rather poorly in some situations that are typically encountered with non-stationary macroeconomic time-series data, so too do the well-known Hodrick-Prescott and Baxter-King filters in these same situations. On the other hand, in those situations where the latter two filters are known to perform well, the new filter performs even better. In the case of stationary data that we have considered, the “fuzzy regression” filter clearly out-performs the Hodrick-Prescott filter. Our empirical examples illustrate that the cycles that are extracted by using this new fuzzy filter can have greater amplitude than those associated with the HP filter, although both filters identify almost identical major turning points in the data.

The choice of the smoothing parameter for the HP filter has attracted considerable attention in the literature. It is known that this choice should take account of the data frequency, and that the extracted cycle is sensitive to the chosen value. In the case of our new FR filter the choice of the number of fuzzy clusters raises corresponding issues. For example, for larger sample sizes it may be appropriate to use three or four fuzzy clusters, rather than just two. This poses no practical problems, as is amply illustrated in the various fuzzy regression examples provided by Giles and Draeseke (2003). In this paper we have focused entirely on quarterly time-series data. Undoubtedly, the frequency of the data at least partly determines the appropriate number of clusters. Our experience with “fuzzy regression” in a wide range of other econometric applications indicates that four or five clusters is the maximum number that is ever needed to successfully model non-linear relationships, even with several thousand observations, and regardless of the data frequency.

Additional Monte Carlo experimentation shows that our results are very robust for sample sizes in the range of 100 to 200 quarterly observations, and to changes in the “fuzzifier” parameter ( $m$ ) that are used in the construction of the “fuzzy regression” filter.<sup>12</sup> The simulation results and empirical applications reported in this paper suggest that the new “fuzzy regression” filter deserves further investigation. Work in progress seeks to explore more fully the circumstances under which this filter performs particularly well, and under which it out-performs the other filters that are in common use.

**Table 1: Monte Carlo Simulation Results  
(Stochastic Trend Model)**

$\sigma_\varepsilon / \sigma_\eta$	Parameters		Cycle Correlation Coefficient		Cycle Std. Deviation Ratio	
	$\theta_1$	$\theta_2$	HP Filter	FR Filter	HP Filter	FR Filter
10	0	0	0.075	0.056	12.646	18.098
			(-0.12,0.27)	(-0.14,0.25)	(8.90,16.40)	(10.55,25.65)
10	1.2	-0.25	0.086	0.100	4.485	6.427
			(-0.21,0.38)	(-0.34,0.54)	(2.02,6.95)	(2.40,10.46)
10	1.2	-0.50	0.122	0.098	6.830	9.753
			(-0.19,0.44)	(-0.23,0.43)	(4.42,9.24)	(5.29,14.22)
10	1.2	-0.75	0.153	0.114	6.351	9.047
			(-0.10,0.40)	(-0.11,0.34)	(4.00,8.70)	(4.78,13.31)
5	0	0	0.149	0.111	6.379	9.090
			(-0.05,0.35)	(-0.09,0.31)	(4.52,8.24)	(5.33,12.85)
5	1.2	-0.25	0.166	0.190	2.292	3.281
			(-0.12,0.45)	(-0.24,0.62)	(1.04,3.54)	(1.25,5.31)
5	1.2	-0.50	0.237	0.192	3.500	4.949
			(-0.07,0.54)	(-0.13,0.52)	(2.30,4.70)	(2.73,7.17)
5	1.2	-0.75	0.294	0.222	3.286	4.608
			(0.05,0.54)	(-0.01,0.45)	(2.14,4.44)	(2.50,6.71)
1	0	0	0.596	0.482	1.589	2.065
			(0.44,0.75)	(0.28,0.68)	(1.26,1.92)	(1.38,2.75)
1	1.2	-0.25	0.532	0.613	0.705	0.985
			(0.29,0.78)	(0.32,0.90)	(0.36,1.05)	(0.50,1.47)
1	1.2	-0.50	0.741	0.680	1.113	1.369
			(0.59,0.90)	(0.47,0.89)	(0.87,1.35)	(0.97,1.77)
1	1.2	-0.75	0.821	0.735	1.153	1.344
			(0.71,0.94)	(0.56,0.91)	(0.97,1.33)	(1.01,1.67)
.5	0	0	0.819	0.731	1.153	1.343
			(0.73,0.91)	(0.58,0.88)	(1.01,1.30)	(1.05,1.63)
.5	1.2	-0.25	0.635	0.739	0.590	0.812
			(0.39,0.88)	(0.50,0.98)	(0.32,0.86)	(0.48,1.15)
.5	1.2	-0.50	0.872	0.859	0.946	1.077
			(0.78,0.97)	(0.75,0.97)	(0.80,1.09)	(0.89,1.27)
.5	1.2	-0.75	0.932	0.899	1.015	1.090
			(0.88,0.98)	(0.82,0.98)	(0.93,1.10)	(0.96,1.22)
.01	0	0	0.979	0.988	0.964	0.987
			(0.95,1.00)	(0.97,1.01)	(0.93,1.00)	(0.96,1.01)
.01	1.2	-0.25	0.685	0.804	0.545	0.744
			(0.44,0.93)	(0.57,1.04)	(0.31,0.78)	(0.47,1.02)
.01	1.2	-0.50	0.935	0.963	0.882	0.959
			(0.86,1.01)	(0.90,1.02)	(0.79,0.98)	(0.89,1.03)
.01	1.2	-0.75	0.981	0.989	0.964	0.989
			(0.96,1.01)	(0.97,1.01)	(0.93,1.00)	(0.97,1.01)

**Table 2: Monte Carlo Simulation Results  
(Stochastic Trend Model)**

$\sigma_\varepsilon / \sigma_\eta$	Parameters		HP Autocorrelations			FR Autocorrelations		
	$\theta_1$	$\theta_2$	$\rho_1$	$\rho_2$	$\rho_3$	$\rho_1$	$\rho_2$	$\rho_3$
10	0	0	0.675 [0.00]	0.419 [0.00]	0.217 [0.00]	0.812	0.649	0.505
			(0.53,0.82)	(0.20,0.64)	(-0.03,0.46)	(0.69,0.94)	(0.44,0.86)	(0.24,0.78)
10	1.2	-0.25	0.681 [0.96]	0.423 [0.90]	0.219 [0.84]	0.816	0.652	0.507
			(0.54,0.83)	(0.21,0.64)	(-0.03,0.47)	(0.69,0.94)	(0.44,0.86)	(0.24,0.78)
10	1.2	-0.50	0.680 [0.80]	0.420 [0.46]	0.213 [0.15]	0.815	0.649	0.502
			(0.54,0.82)	(0.20,0.64)	(-0.04,0.46)	(0.69,0.94)	(0.44,0.86)	(0.23,0.77)
10	1.2	-0.75	0.679 [0.69]	0.412 [0.07]	0.201 [-0.42]	0.813	0.644	0.493
			(0.54,0.82)	(0.19,0.63)	(-0.05,0.45)	(0.69,0.94)	(0.43,0.86)	(0.22,0.77)
5	0	0	0.661 [0.00]	0.411 [0.00]	0.212 [0.00]	0.804	0.643	0.500
			(0.51,0.81)	(0.19,0.63)	(-0.03,0.46)	(0.67,0.94)	(0.43,0.86)	(0.23,0.77)
5	1.2	-0.25	0.685 [0.96]	0.426 [0.90]	0.220 [0.84]	0.818	0.653	0.507
			(0.54,0.83)	(0.21,0.65)	(-0.03,0.47)	(0.69,0.94)	(0.44,0.86)	(0.24,0.78)
5	1.2	-0.50	0.683 [0.80]	0.413 [0.46]	0.199 [0.15]	0.813	0.641	0.488
			(0.54,0.82)	(0.20,0.63)	(-0.05,0.45)	(0.69,0.94)	(0.43,0.85)	(0.21,0.76)
5	1.2	-0.75	0.678 [0.69]	0.385 [0.07]	0.154 [-0.42]	0.808	0.621	0.457
			(0.54,0.81)	(0.17,0.61)	(-0.11,0.42)	(0.69,0.93)	(0.40,0.84)	(0.16,0.75)
1	0	0	0.394 [0.00]	0.237 [0.00]	0.114 [0.00]	0.609	0.487	0.380
			(0.18,0.61)	(0.01,0.46)	(-0.11,0.33)	(0.38,0.84)	(0.22,0.75)	(0.10,0.66)
1	1.2	-0.25	0.736 [0.96]	0.464 [0.90]	0.238 [0.84]	0.843	0.669	0.509
			(0.61,0.86)	(0.25,0.68)	(-0.02,0.49)	(0.74,0.95)	(0.47,0.86)	(0.25,0.77)
1	1.2	-0.50	0.710 [0.80]	0.338 [0.46]	0.029 [0.15]	0.791	0.516	0.275
			(0.61,0.81)	(0.15,0.53)	(-0.21,0.27)	(0.69,0.89)	(0.31,0.73)	(-0.02,0.57)
1	1.2	-0.75	0.664 [0.69]	0.135 [0.07]	-0.281 [-0.42]	0.738	0.320	-0.018
			(0.59,0.74)	(-0.05,0.31)	(-0.53,-0.03)	(0.64,0.83)	(0.07,0.57)	(-0.38,0.35)
.5	0	0	0.145 [0.00]	0.072 [0.00]	0.020 [0.00]	0.346	0.275	0.215
			(-0.07,0.36)	(-0.13,0.28)	(-0.17,0.21)	(0.08,0.61)	(0.01,0.54)	(-0.05,0.48)
.5	1.2	-0.25	0.759 [0.96]	0.482 [0.90]	0.246 [0.84]	0.854	0.675	0.508
			(0.65,0.87)	(0.28,0.69)	(-0.01,0.50)	(0.76,0.95)	(0.49,0.87)	(0.25,0.77)
.5	1.2	-0.50	0.720 [0.80]	0.306 [0.46]	-0.043 [0.15]	0.775	0.434	0.137
			(0.63,0.81)	(0.12,0.49)	(-0.28,0.19)	(0.69,0.86)	(0.24,0.63)	(-0.14,0.41)
.5	1.2	-0.75	0.658 [0.69]	0.052 [0.07]	-0.423 [-0.42]	0.697	0.155	-0.275
			(0.59,0.72)	(-0.10,0.20)	(-0.63,-0.22)	(0.63,0.77)	(-0.04,0.35)	(-0.56,0.01)
.01	0	0	-0.082 [0.00]	-0.080 [0.00]	-0.068 [0.00]	-0.036	-0.039	-0.033
			(-0.27,0.10)	(-0.27,0.11)	(-0.26,0.12)	(-0.23,0.16)	(-0.23,0.15)	(-0.23,0.16)
.01	1.2	-0.25	0.772 [0.96]	0.491 [0.90]	0.250 [0.84]	0.860	0.677	0.505
			(0.67,0.88)	(0.29,0.70)	(-0.01,0.51)	(0.77,0.95)	(0.49,0.87)	(0.24,0.77)
.01	1.2	-0.50	0.726 [0.80]	0.288 [0.46]	-0.081 [0.15]	0.762	0.377	0.042
			(0.64,0.81)	(0.11,0.47)	(-0.31,0.15)	(0.68,0.84)	(0.19,0.56)	(-0.21,0.30)
.01	1.2	-0.75	0.655 [0.69]	0.011 [0.07]	-0.492 [-0.42]	0.669	0.049	-0.439
			(0.59,0.72)	(-0.13,0.15)	(-0.67,-0.31)	(0.61,0.73)	(-0.10,0.20)	(-0.64,-0.24)

**Table 3: Monte Carlo Simulation Results  
(Stochastic Trend Model)**

$\sigma_\varepsilon/\sigma_\eta$	Parameters		HP Partial Autocorrelations			FR Partial Autocorrelations		
	$\theta_1$	$\theta_2$	$\rho_1^*$	$\rho_2^*$	$\rho_3^*$	$\rho_1^*$	$\rho_2^*$	$\rho_3^*$
10	0	0	0.675 [0.00]	-0.080 [0.00]	-0.077 [0.00]	0.812	-0.047	-0.048
			(0.53,0.82)	(-0.26,0.10)	(-0.26,0.11)	(0.69,0.94)	(-0.22,0.13)	(-0.23,0.13)
10	1.2	-0.25	0.681 [0.96]	-0.088 [-0.25]	-0.078 [0.00]	0.816	-0.056	-0.048
			(0.54,0.83)	(-0.27,0.09)	(-0.26,0.11)	(0.69,0.94)	(-0.23,0.12)	(-0.23,0.13)
10	1.2	-0.50	0.680 [0.80]	-0.093 [-0.50]	-0.080 [0.00]	0.815	-0.060	-0.049
			(0.54,0.82)	(-0.27,0.09)	(-0.26,0.10)	(0.69,0.94)	(-0.24,0.12)	(-0.23,0.13)
10	1.2	-0.75	0.679 [0.69]	-0.104 [-0.75]	-0.083 [0.00]	0.813	-0.068	-0.049
			(0.54,0.82)	(-0.29,0.08)	(-0.27,0.10)	(0.69,0.94)	(-0.25,0.11)	(-0.23,0.13)
5	0	0	0.661 [0.00]	-0.061	-0.077 [0.00]	0.804	-0.027	-0.047
			(0.51,0.81)	(-0.24,0.12)	(-0.26,0.11)	(0.67,0.94)	(-0.20,0.15)	(-0.23,0.13)
5	1.2	-0.25	0.685 [0.96]	-0.094 [-0.25]	-0.079 [0.00]	0.818	-0.062	-0.050
			(0.54,0.83)	(-0.27,0.09)	(-0.26,0.10)	(0.69,0.94)	(-0.24,0.11)	(-0.23,0.13)
5	1.2	-0.50	0.683 [0.80]	-0.111 [-0.50]	-0.086 [0.00]	0.813	-0.076	-0.051
			(0.54,0.82)	(-0.29,0.07)	(-0.27,0.10)	(0.69,0.94)	(-0.25,0.10)	(-0.23,0.13)
5	1.2	-0.75	0.678 [0.69]	-0.150 [-0.75]	-0.097 [0.00]	0.808	-0.107	-0.050
			(0.54,0.81)	(-0.34,0.04)	(-0.28,0.09)	(0.69,0.93)	(-0.29,0.08)	(-0.23,0.13)
1	0	0	0.394 [0.00]	0.084 [0.00]	-0.014 [0.00]	0.609	0.171	0.036
			(0.18,0.61)	(-0.10,0.27)	(-0.20,0.17)	(0.38,0.84)	(-0.01,0.35)	(-0.14,0.22)
1	1.2	-0.25	0.736 [0.96]	-0.182 [-0.25]	-0.088 [0.00]	0.843	-0.157	-0.060
			(0.61,0.86)	(-0.36,-0.00)	(-0.27,0.09)	(0.74,0.95)	(-0.33,0.02)	(-0.24,0.12)
1	1.2	-0.50	0.710 [0.80]	-0.344 [-0.50]	-0.125 [0.00]	0.791	-0.303	-0.063
			(0.61,0.81)	(-0.53,-0.16)	(-0.31,0.06)	(0.69,0.89)	(-0.49,-0.11)	(-0.25,0.12)
1	1.2	-0.75	0.664 [0.69]	-0.555 [-0.75]	-0.139 [0.00]	0.738	-0.496	-0.021
			(0.59,0.74)	(-0.74,-0.37)	(-0.33,0.05)	(0.64,0.83)	(-0.70,-0.30)	(-0.22,0.18)
.5	0	0	0.15 [0.00]	0.04 [0.00]	0.00 [0.00]	0.35	0.16	0.08
			(-0.07,0.36)	(-0.15,0.24)	(-0.19,0.19)	(0.08,0.61)	(-0.04,0.37)	(-0.10,0.27)
.5	1.2	-0.25	0.759 [0.96]	-0.237 [-0.25]	-0.084 [0.00]	0.854	-0.215	-0.056
			(0.65,0.87)	(-0.42,-0.06)	(-0.26,0.10)	(0.76,0.95)	(-0.39,-0.04)	(-0.23,0.12)
.5	1.2	-0.50	0.720 [0.80]	-0.452 [-0.50]	-0.109 [0.00]	0.775	-0.422	-0.052
			(0.63,0.81)	(-0.62,-0.28)	(-0.29,0.07)	(0.69,0.86)	(-0.60,-0.24)	(-0.24,0.13)
.5	1.2	-0.75	0.658 [0.69]	-0.678 [-0.75]	-0.116 [0.00]	0.697	-0.644	-0.019
			(0.59,0.72)	(-0.83,-0.53)	(-0.30,0.07)	(0.63,0.77)	(-0.81,-0.48)	(-0.22,0.18)
.01	0	0	-0.082 [0.00]	-0.097 [0.00]	-0.084 [0.00]	-0.036	-0.050	-0.037
			(-0.27,0.10)	(-0.28,0.09)	(-0.27,0.10)	(-0.23,0.16)	(-0.24,0.14)	(-0.23,0.16)
.01	1.2	-0.25	0.772 [0.96]	-0.274 [-0.25]	-0.076 [0.00]	0.860	-0.253	-0.050
			(0.67,0.88)	(-0.45,-0.10)	(-0.26,0.10)	(0.77,0.95)	(-0.43,-0.08)	(-0.23,0.13)
.01	1.2	-0.50	0.726 [0.80]	-0.512 [-0.50]	-0.088 [0.00]	0.762	-0.495	-0.044
			(0.64,0.81)	(-0.67,-0.35)	(-0.27,0.09)	(0.68,0.84)	(-0.66,-0.33)	(-0.23,0.14)
.01	1.2	-0.75	0.655 [0.69]	-0.736 [-0.75]	-0.093 [0.00]	0.669	-0.726	-0.047
			(0.59,0.72)	(-0.87,-0.61)	(-0.27,0.09)	(0.61,0.73)	(-0.86,-0.59)	(-0.23,0.14)

**Table 4: Monte Carlo Simulation Results  
(Deterministic Trend Model)**

$\sigma_\varepsilon / \sigma_\eta$	Parameters		Cycle Correlation Coefficient		Cycle Std. Deviation Ratio	
	$\theta_1$	$\theta_2$	HP Filter	FR Filter	HP Filter	FR Filter
10	0	0	0.090	0.090	9.600	9.866
			(-1.33, 0.31)	(-1.30, 0.31)	(7.30, 11.90)	(7.51, 12.22)
10	1.2	-0.25	0.126	0.192	3.566	3.700
			(-0.04, 0.30)	(-0.03, 0.42)	(1.53, 5.61)	(1.61, 5.80)
10	1.2	-0.50	0.168	0.181	5.241	5.395
			(-0.04, 0.38)	(-0.04, 0.41)	(3.47, 7.00)	(3.59, 7.19)
10	1.2	-0.75	0.204	0.204	4.831	4.963
			(-0.02, 0.42)	(-0.02, 0.42)	(3.22, 6.43)	(3.32, 6.59)
5	0	0	0.186	0.187	4.868	5.002
			(-0.03, 0.40)	(-0.02, 0.40)	(3.71, 6.02)	(3.82, 6.18)
5	1.2	-0.25	0.239	0.355	1.855	1.975
			(0.05, 0.42)	(0.12, 0.58)	(0.81, 2.89)	(0.91, 3.03)
5	1.2	-0.50	0.318	0.341	2.736	2.828
			(0.10, 0.53)	(0.12, 0.56)	(1.87, 3.60)	(1.94, 3.70)
5	1.2	-0.75	0.381	0.382	2.561	2.629
			(0.16, 0.59)	(0.16, 0.59)	(1.78, 3.34)	(1.83, 3.42)
1	0	0	0.698	0.701	1.354	1.389
			(0.58, 0.81)	(0.58, 0.81)	(1.12, 1.58)	(1.15, 1.62)
1	1.2	-0.25	0.479	0.612	0.492	0.660
			(0.17, 0.78)	(0.21, 1.00)	(0.15, 0.82)	(0.20, 1.11)
1	1.2	-0.50	0.825	0.863	1.002	1.087
			(0.70, 0.94)	(0.76, 0.96)	(0.82, 1.18)	(0.92, 1.25)
1	1.2	-0.75	0.891	0.897	1.071	1.099
			(0.81, 0.96)	(0.82, 0.96)	(0.94, 1.19)	(0.97, 1.22)
.5	0	0	0.882	0.888	1.076	1.101
			(0.83, 0.93)	(0.83, 0.93)	(0.95, 1.19)	(0.98, 1.21)
.5	1.2	-0.25	0.709	0.844	0.605	0.804
			(0.45, 0.96)	(0.64, 1.04)	(0.32, 0.89)	(0.48, 1.12)
.5	1.2	-0.50	0.914	0.940	0.935	1.007
			(0.83, 0.99)	(0.88, 0.99)	(0.81, 1.05)	(0.90, 1.11)
.5	1.2	-0.75	0.958	0.964	1.000	1.023
			(0.92, 0.99)	(0.93, 0.99)	(0.93, 1.06)	(0.95, 1.08)
.01	0	0	0.984	0.992	0.969	0.991
			(0.96, 1.00)	(0.97, 1.00)	(0.93, 1.00)	(0.96, 1.01)
.01	1.2	-0.25	0.742	0.867	0.577	0.782
			(0.48, 0.99)	(0.67, 1.06)	(0.31, 0.84)	(0.47, 1.09)
.01	1.2	-0.50	0.951	0.975	0.898	0.971
			(0.87, 1.02)	(0.92, 1.02)	(0.78, 1.00)	(0.88, 1.05)
.01	1.2	-0.75	0.987	0.993	0.970	0.992
			(0.96, 1.00)	(0.97, 1.00)	(0.93, 1.00)	(0.96, 1.01)

**Table 5: Monte Carlo Simulation Results  
(Deterministic Trend Model)**

$\sigma_\varepsilon / \sigma_\eta$	Parameters		HP Autocorrelations			FR Autocorrelations		
	$\theta_1$	$\theta_2$	$\rho_1$	$\rho_2$	$\rho_3$	$\rho_1$	$\rho_2$	$\rho_3$
10	0	0	0.023 [0.00]	-0.082 [0.00]	-0.074 [0.00]	0.072	-0.031	-0.030
			(-0.20, 0.25)	(-0.29, 0.13)	(-0.28, 0.14)	(-0.16, 0.30)	(-0.24, 0.18)	(-0.24, 0.18)
10	1.2	-0.25	0.049 [0.96]	-0.062 [0.90]	-0.062 [0.84]	0.113	0.006	0.000
			(-0.18, 0.28)	(-0.27, 0.15)	(-0.26, 0.14)	(-0.13, 0.35)	(-0.20, 0.21)	(-0.20, 0.20)
10	1.2	-0.50	0.047 [0.80]	-0.069 [0.46]	-0.074 [0.15]	0.098	-0.016	-0.027
			(-0.18, 0.27)	(-0.28, 0.14)	(-0.28, 0.14)	(-0.13, 0.33)	(-0.23, 0.19)	(-0.24, 0.18)
10	1.2	-0.75	0.052 [0.69]	-0.078[0.07]	-0.093 [-0.42]	0.099	-0.028	-0.049
			(-0.17, 0.27)	(-0.28, 0.12)	(-0.30, 0.12)	(-0.12, 0.32)	(-0.23, 0.17)	(-0.26, 0.16)
5	0	0	0.021 [0.00]	-0.082 [0.00]	-0.074 [0.00]	0.070	-0.031	-0.030
			(-0.20, 0.24)	(-0.29, 0.13)	(-0.28, 0.13)	(-0.15, 0.29)	(-0.24, 0.18)	(-0.24, 0.18)
5	1.2	-0.25	0.108 [0.96]	-0.015 [0.90]	-0.032 [0.84]	0.211	0.097	0.072
			(-0.12, 0.34 )	(-0.23, 0.20)	(-0.23, 0.16)	(-0.03, 0.46)	(-0.12, 0.32)	(-0.14, 0.28)
5	1.2	-0.50	0.106 [0.80]	-0.037 [0.46]	-0.072 [0.15]	0.160	0.021	-0.018
			(-0.12, 0.33)	(-0.25, 0.17)	(-0.27, 0.13)	(-0.07, 0.39)	(-0.19, 0.23)	(-0.22, 0.19)
5	1.2	-0.75	0.120 [0.69]	-0.068 [0.07]	-0.138 [-0.42]	0.163	-0.020	-0.093
			(-0.10, 0.34)	(-0.26, 0.13)	(-0.35, 0.07)	(-0.05, 0.38)	(-0.22, 0.18)	(-0.31, 0.12)
1	0	0	-0.024 [0.00]	-0.077 [0.00]	-0.067 [0.00]	0.023	-0.031	-0.027
			(-0.23, 0.19)	(-0.29, 0.14)	(-0.29, 0.16)	(-0.19, 0.24)	(-0.24, 0.18)	(-0.25, 0.20)
1	1.2	-0.25	0.579 [0.96]	0.363 [0.90]	0.205 [0.84]	0.749	0.605	0.485
			(0.37, 0.78)	(0.11, 0.61)	(-0.05, 0.46)	(0.57, 0.92)	(0.36, 0.84)	(0.20, 0.76)
1	1.2	-0.50	0.589 [0.80]	0.264 [0.46]	0.000 [0.15]	0.646	0.357	0.115
			(0.42, 0.75)	(0.05, 0.47)	(-0.22, 0.22)	(0.49, 0.79)	(0.15, 0.56)	(-0.13, 0.36)
1	1.2	-0.75	0.562 [0.69]	0.049 [0.07]	-0.359 [-0.42]	0.581	0.088	-0.308
			(0.43, 0.68)	(-0.10, 0.20)	(-0.54, -0.17)	(0.46, 0.70)	(-0.06, 0.24)	(-0.51, -0.10)
.5	0	0	-0.054 [0.00]	-0.075 [0.00]	-0.063 [0.00]	-0.009	-0.032	-0.026
			(-0.27, 0.16)	(-0.29, 0.14)	(-0.28, 0.16)	(-0.23, 0.21)	(-0.24, 0.17)	(-0.25, 0.19)
.5	1.2	-0.25	0.718 [0.96]	0.464 [0.90]	0.254 [0.84]	0.829	0.661	0.508
			(0.56, 0.86)	(0.23, 0.69)	(-0.02, 0.52)	(0.71, 0.94)	(0.45, 0.86)	(0.23, 0.78)
.5	1.2	-0.50	0.673 [0.80]	0.268 [0.46]	-0.062 [0.15]	0.713	0.352	0.050
			(0.55, 0.79)	(0.07, 0.46)	(-0.29, 0.17)	(0.60, 0.82)	(0.15, 0.54)	(-0.21, 0.31)
.5	1.2	-0.75	0.618 [0.69]	0.009 [0.07]	-0.455 [-0.42]	0.632	0.044	-0.407
			(0.53, 0.70)	(-0.14, 0.15)	(-0.64, -0.26)	(0.54, 0.71)	(-0.10, 0.19)	(-0.61, -0.20)
.01	0	0	-0.076 [0.00]	-0.074 [0.00]	-0.063 [0.00]	-0.032	-0.034	-0.028
			(-0.30, 0.15)	(-0.28, 0.14)	(-0.28, 0.15)	(-0.26, 0.20)	(-0.25, 0.18)	(-0.24, 0.19)
.01	1.2	-0.25	0.780 [0.96]	0.510 [0.90]	0.280 [0.84]	0.868	0.696	0.535
			(0.66, 0.89)	(0.28, 0.73)	(-0.01, 0.57)	(0.77, 0.96)	(0.49, 0.89)	(0.25, 0.81)
.01	1.2	-0.50	0.724 [0.80]	0.295 [0.46]	-0.062 [0.15]	0.759	0.378	0.054
			(0.62, 0.82)	(0.10, 0.49)	(-0.31, 0.18)	(0.66, 0.84)	(0.17, 0.58)	(-0.22, 0.33)
.01	1.2	-0.75	0.652 [0.69]	0.014 [0.07]	-0.476 [-0.42]	0.664	0.049	-0.427
			(0.58, 0.72)	(-0.13, 0.16)	(-0.66, -0.28)	(0.59, 0.73)	(-0.11, 0.20)	(-0.63, -0.21)

**Table 6: Monte Carlo Simulation Results  
(Deterministic Trend Model)**

$\sigma_\varepsilon/\sigma_\eta$	Parameters		HP Partial Autocorrelations			FR Partial Autocorrelations		
	$\theta_1$	$\theta_2$	$\rho_1^*$	$\rho_2^*$	$\rho_3^*$	$\rho_1^*$	$\rho_2^*$	$\rho_3^*$
10	0	0	0.023 [0.00]	-0.097 [0.00]	-0.068 [0.00]	0.072	-0.052	-0.022
			(-0.20, 0.25)	(-0.31, 0.11)	(-0.26, 0.13)	(-0.16, 0.30)	(-0.27, 0.17)	(-0.22, 0.18)
10	1.2	-0.25	0.049 [0.96]	-0.080 [-0.25]	-0.053 [0.00]	0.113	-0.022	0.003
			(-0.19, 0.27)	(-0.29, 0.13)	(-0.24, 0.13)	(-0.12, 0.35)	(-0.23, 0.18)	(-0.19, 0.20)
10	1.2	-0.50	0.047 [0.80]	-0.087 [-0.50]	-0.065 [0.00]	0.098	-0.041	-0.020
			(-0.18, 0.27)	(-0.30, 0.13)	(-0.26, 0.13)	(-0.13, 0.33)	(-0.25, 0.17)	(-0.22, 0.18)
10	1.2	-0.75	0.052 [0.69]	-0.096 [-0.75]	-0.083 [0.00]	0.099	-0.053	-0.039
			(-0.17, 0.27)	(-0.31, 0.11)	(-0.28, 0.11)	(-0.12, 0.32)	(-0.26, 0.16)	(-0.24, 0.16)
5	0	0	0.021 [0.00]	-0.096[0.00]	-0.068 [0.00]	0.070	-0.051	-0.022
			(-0.20, 0.24)	(-0.31, 0.12)	(-0.26, 0.13)	(-0.15, 0.29)	(-0.26, 0.16)	(-0.22, 0.18)
5	1.2	-0.25	0.108 [0.96]	-0.042 [-0.25]	-0.027 [0.00]	0.211	0.038	0.044
			(-0.12, 0.34)	(-0.26, 0.17)	(-0.22, 0.16)	(-0.03, 0.46)	(-0.17, 0.25)	(-0.15, 0.24)
5	1.2	-0.50	0.106 [0.80]	-0.064 [-0.50]	-0.062 [0.00]	0.160	-0.019	-0.019
			(-0.12, 0.33)	(-0.28, 0.15)	(-0.26, 0.13)	(-0.07, 0.39)	(-0.23, 0.19)	(-0.22, 0.18)
5	1.2	-0.75	0.120 [0.69]	-0.099 [-0.75]	-0.120 [0.00]	0.163	-0.063	-0.081
			(-0.10, 0.34)	(-0.31, 0.11)	(-0.32, 0.08)	(-0.05, 0.38)	(-0.27, 0.15)	(-0.29, 0.13)
1	0	0	-0.024 [0.00]	-0.091 [0.00]	-0.074 [0.00]	0.023	-0.045	-0.027
			(-0.23, 0.19)	(-0.31, 0.13)	(-0.28, 0.13)	(-0.19, 0.24)	(-0.26, 0.17)	(-0.25, 0.19)
1	1.2	-0.25	0.579 [0.96]	0.023 [-0.25]	-0.038 [0.00]	0.749	0.081	-0.001
			(0.37, 0.78)	(-0.17, 0.21)	(-0.26, 0.18)	(0.57, 0.92)	(-0.11, 0.27)	(-0.22, 0.22)
1	1.2	-0.50	0.589 [0.80]	-0.146 [-0.50]	-0.160 [0.00]	0.646	-0.120	-0.128
			(0.42, 0.75)	(-0.34, 0.05)	(-0.37, 0.05)	(0.49, 0.79)	(-0.32, 0.08)	(-0.35, 0.09)
1	1.2	-0.75	0.562 [0.69]	-0.405 [-0.75]	-0.303 [0.00]	0.581	-0.393	-0.274
			(0.43, 0.68)	(-0.63, -0.18)	(-0.48, -0.11)	(0.46, 0.70)	(-0.62, -0.16)	(-0.46, -0.08)
.5	0	0	-0.054 [0.00]	-0.092 [0.00]	-0.075 [0.00]	-0.009	-0.046	-0.029
			(-0.27, 0.16)	(-0.31, 0.12)	(-0.28, 0.13)	(-0.23, 0.21)	(-0.26, 0.16)	(-0.24, 0.18)
.5	1.2	-0.25	0.718 [0.96]	-0.127 [-0.25]	-0.084 [0.00]	0.829	-0.103	-0.056
			(0.56, 0.86)	(-0.32, 0.07)	(-0.30, 0.13)	(0.71, 0.94)	(-0.31, 0.10)	(-0.27, 0.15)
.5	1.2	-0.50	0.673 [0.80]	-0.352 [-0.50]	-0.159 [0.00]	0.713	-0.331	-0.122
			(0.55, 0.79)	(-0.55, -0.15)	(-0.36, 0.04)	(0.60, 0.82)	(-0.53, -0.12)	(-0.33, 0.08)
.5	1.2	-0.75	0.618 [0.69]	-0.611 [-0.75]	-0.229 [0.00]	0.632	-0.599	-0.193
			(0.53, 0.70)	(-0.79, -0.43)	(-0.41, -0.04)	(0.54, 0.71)	(-0.78, -0.41)	(-0.38, 0.00)
.01	0	0	-0.076 [0.00]	-0.095 [0.00]	-0.077 [0.00]	-0.032	-0.050	-0.032
			(-0.30, 0.15)	(-0.30, 0.11)	(-0.27, 0.12)	(-0.26, 0.20)	(-0.26, 0.16)	(-0.24, 0.18)
.01	1.2	-0.25	0.780 [0.96]	-0.266 [-0.25]	-0.065 [0.00]	0.868	-0.251	-0.036
			(0.66, 0.89)	(-0.48, -0.05)	(-0.27, 0.14)	(0.77, 0.96)	(-0.46, -0.03)	(-0.24, 0.17)
.01	1.2	-0.50	0.724 [0.80]	-0.495 [-0.50]	-0.079 [0.00]	0.759	-0.475	-0.039
			(0.62, 0.82)	(-0.68, -0.30)	(-0.27, 0.12)	(0.66, 0.84)	(-0.67, -0.27)	(-0.24, 0.16)
.01	1.2	-0.75	0.652 [0.69]	-0.720 [-0.75]	-0.090 [0.00]	0.664	-0.708	-0.050
			(0.58, 0.72)	(-0.86, -0.57)	(-0.30, 0.12)	(0.59, 0.73)	(-0.85, -0.56)	(-0.26, 0.16)



**Table 7: Empirical Applications**

<b>a. Unit Root Tests</b>									
	T	ADF Tests			KPSS tests				
		Drift	Drift+	Trend	Level	Trend			
<b>Real GDP</b>	206	-0.560		-2.534		1.806*		0.266*	
<b>Real Durables Consumption</b>	100	-0.473		-2.739		1.176*		0.172**	
<b>Capacity Utilization</b>	158	-4.235*		-4.222*		0.109		0.102	

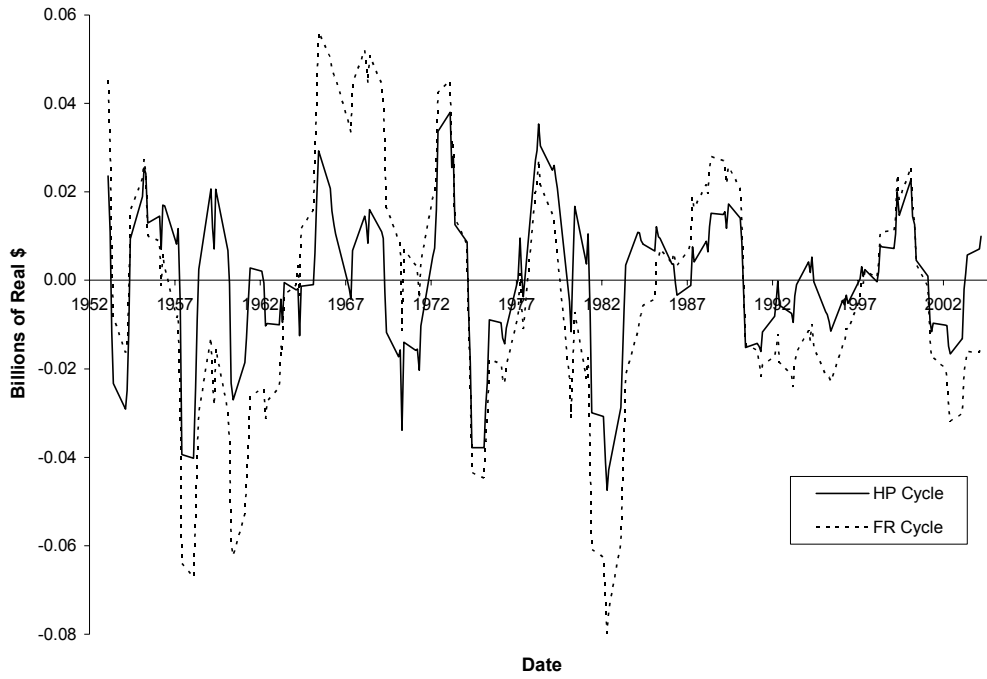
  

<b>b. Cycle Characteristics</b>										
	T	c.v. (%)			# Peaks			# Troughs		
		HP	FR	FR	HP	FR	FR	HP	FR	FR
			(c=2)	(c=3)		(c=2)	(c=3)		(c=2)	(c=3)
<b>Real GDP</b>	0.618	0.599	0.595	44	36	36	45	37	37	
<b>Real Durables Consumption</b>	1.310	1.389	n.a.	20	21	n.a.	21	22	n.a.	
<b>Capacity Utilization</b>	0.835	0.800	n.a.	30	23	n.a.	29	22	n.a.	

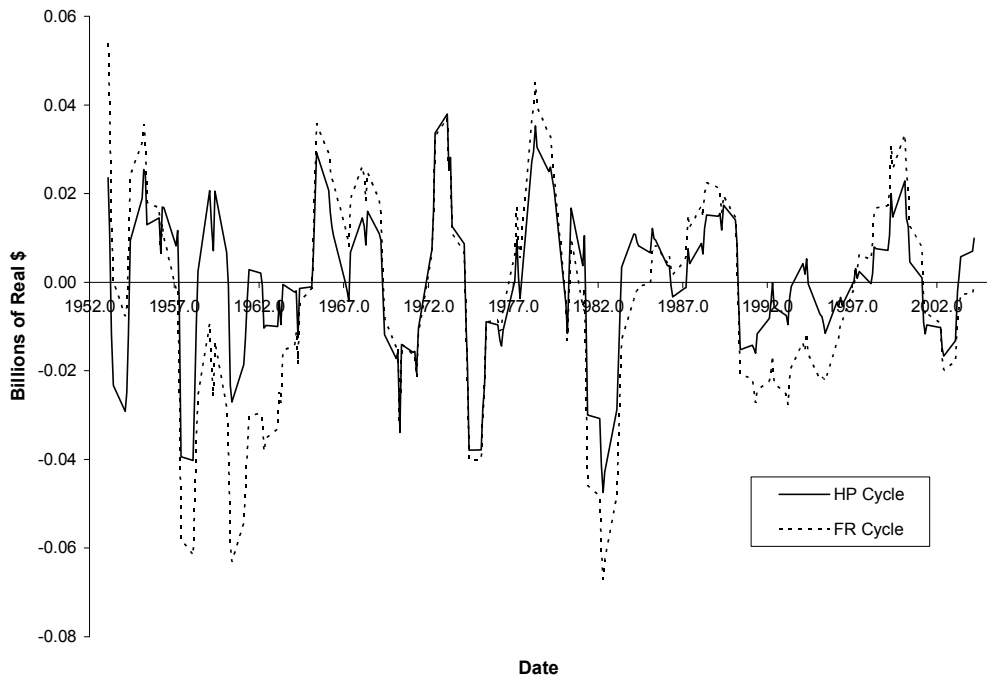
\* Significant at the 1% level

\*\* Significant at the 5% level

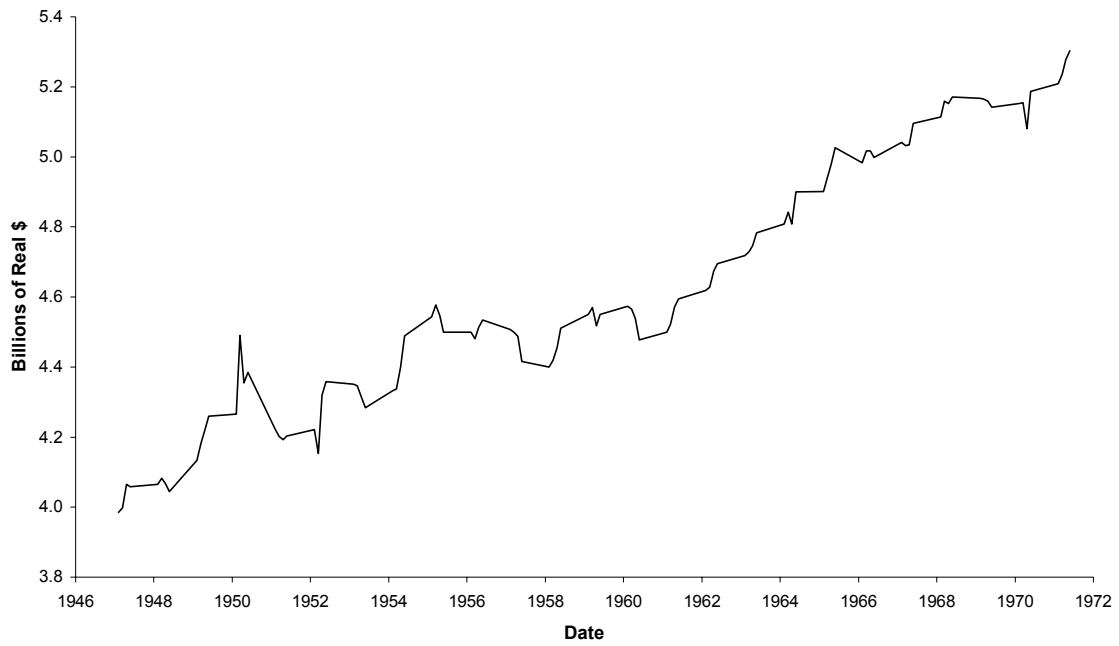
**Figure 1a: Logarithm of U.S. Real GDP  
Extracted Cycles (c = 2)**



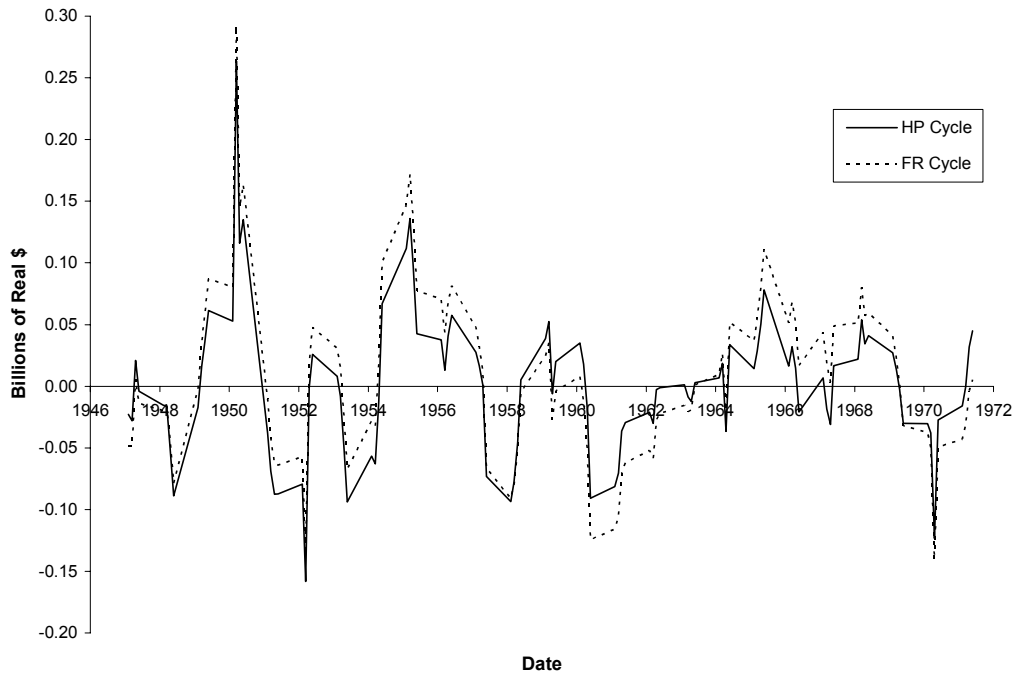
**Figure 1b: Logarithm of U.S. Real GDP  
Extracted Cycles (c = 3)**



**Figure 2a: Logarithm of Durables Consumption Expenditure  
(U.S., 1947Q1 - 1971Q4)**



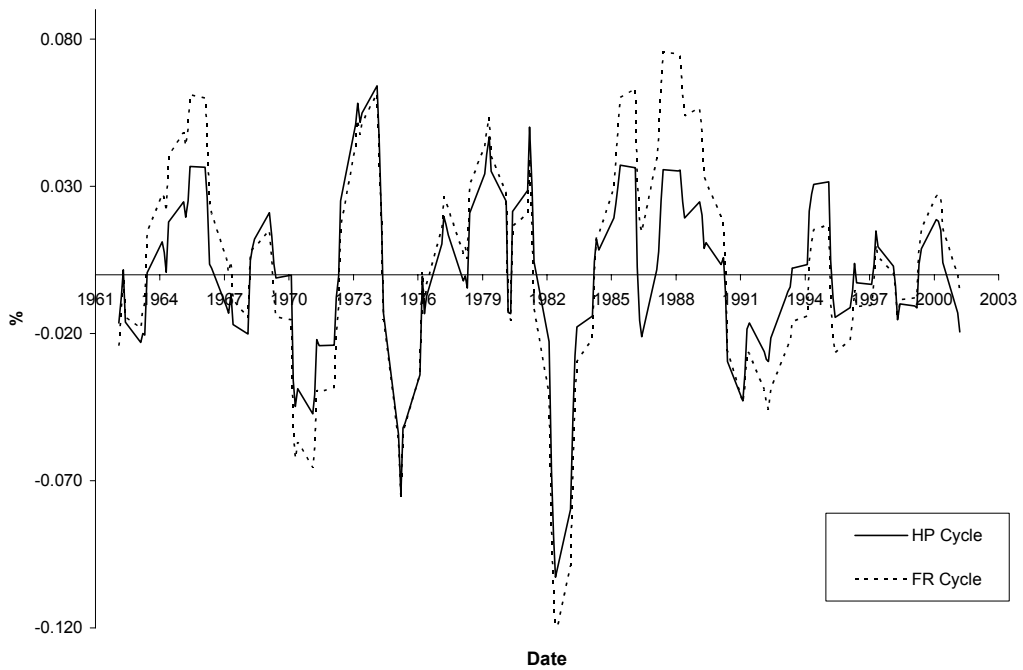
**Figure 2b: Logarithm of U.S. Durables Expenditure  
Extracted Cycles**



**Figure 3a: Logarithm of Capacity Utilization  
(Canada, 1962Q1 - 2001Q2)**



**Figure 3b: Logarithm of Canadian Capacity Utilization  
Extracted Cycles**



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## Footnotes

- \* An earlier version of this paper was presented at the Joint Statistical Meetings, Toronto, in August 2004. We are grateful for the participants' helpful comments and suggestions.
1. This algorithm is a generalization of the "hard" c-means algorithm, and it appears to date from Ruspini (1970). The FCM algorithm is closely associated with such early contributors as Bezdek (1973) and Dunn (1974, 1977), and is widely used in such fields as pattern recognition, for example. Generally, the number of clusters is chosen in advance, and this is the strategy that we adopt.
  2. See Bezdek (1981, Chapter 3) for complete and more formal mathematical details.
  3. The extension of this discussion to the case of several explanatory variables is provided by Giles and Draeseke (2003), but is not needed for our purpose here. There is no need to make any distributional assumptions about the error, but if this is done then these can be taken into account in the subsequent analysis. If the disturbance has a zero mean, the fuzzy function represents the conditional mean of the output variable,  $y$ .
  4. For other recent econometric applications using fuzzy clustering see Giles (2004) and Giles and Hui (2004).
  5. We also replicated the entire analysis with  $T = 200$  and  $c = 5$ . None of the following conclusions were altered, but the results of this expended experiment are available from the authors on request.
  6. See also Levy and Dezhbakhsh (2003).
  7. Source: U.S. Department of Commerce (Bureau of Economic Analysis), series GDPC96 (downloaded, 17 December 2004).
  8. Usually, we define the coefficient of variation as  $c.v. = (\sigma_y / \bar{y})$ . However this can yield misleading results if some of the data are negative. Accordingly, we use the definition  $c.v. = (\sigma_y / \sum_i |y_i|)$ .
  9. Of course, with  $c = 1$  the fitted fuzzy regression is a straight line.
  10. Source: U.S. Department of Commerce (Bureau of Economic Analysis), series PCDGCC96 (downloaded, 17 December 2004).
  11. Source: Cansim II Database, Statistics Canada, series V142812 (downloaded, 17 December 2004).
  12. Details are available from the authors, on request.