Department of Mathematics

MATH 222 - Discrete and Combinatorial Mathematics

Answers and Hints to Practice Questions

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Acknowledgements

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- Kieka Mynhardt's notes, assignments, and tests for Math 222
- Introduction to Combinatorics and Graph Theory - Custom Edition for the University of Victoria
- Discrete Mathematics: Study Guide for MAT212-S - Dr. Kieka Myndardt
- Discrete Mathematics - Norman L. Biggs
- Applied Combinatorics, fourth edition - Alan Tucker
- Discrete Mathematics, An Introduction to Mathematical Reasoning - Susanna S. Epp
- Discrete Mathematics with Combinatorics - James A. Anderson

Contents

1 Preliminaries

1.1 Sets

Solutions:

1. *Hint*: Prove that $A \subseteq B$ and $B \subseteq A$.

2. (a) $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{x\}, \{y\}, \{1,2\}, \{1,3\}, \{1,x\}, \{1,y\}, \{2,3\}, \{2,x\}, \{2,y\}, \{3,x\}, \{3,y\}, \{x,y\}, \{1,2,3\}, \{1,2,x\}, \{1,2,y\}, \{1,3,x\}, \{1,3, y\}, \{1,x,y\}, \{2,3,x\}, \{2,3,y\}, \{2,x,y\}, \{3,x,y\}, \{1,2,3,x\}, \{1,2,3,y\}, \{2,3, x,y\}, \{1,3,x,y\}, \{1,2,x,y\}, A \cup B\}$.

   (b) $\mathcal{P}(B \times C) = \{\emptyset, \{(x,u)\}, \{(x,v)\}, \{(y,u)\}, \{(y,v)\}, \{(x,u),(x,v)\}, \{(x,u), (y,u)\}, \{(x,u),(y,v)\}, \{(x,v),(y,u)\}, \{(x,v),(y,v)\}, \{(y,u),(y,v)\}, \{(x,u), (x,v),(y,v)\}, \{(x,u),(x,v),(y,v)\}, \{(x,u),(y,u),(y,v)\}, \{(x,v),(y,u),(y,v)\}, B \times C\}$.

   (c) $\mathcal{P}(\mathcal{P}(C)) = \{\emptyset, \{\emptyset\}, \{\{u\}\}, \{\{v\}\}, \{C\}, \{\emptyset, \{u\}\}, \{\emptyset, \{v\}\}, \{\emptyset, C\}, \{\{u\}, \{v\}\}, \{\{u\}, C\}, \{\{v\}, C\}, \{\emptyset, \{u\}, \{v\}\}, \{\emptyset, \{u\}, C\}, \{\emptyset, \{v\}, C\}, \{\{u\}, \{v\}, C\}, \{\emptyset, \{u\}, \{v\}, C\}\}$.

   (d) $A \times (B \cap C) = A \times \emptyset = \emptyset$.

   (e) $(A \times B) \times C = \{((1,x),u), ((1,y),u), ((2,x),u), ((2,y),u), ((3,x),u), ((3, y),u), ((1,x),v), ((1,y),v), ((2,x),v), ((2,y),v), ((3,x),v), ((3,y),v)\}$.

3. *Hint*: Proof by contradiction.

4. *Hint*: Consider an arbitrary element of A.

5. *Hint*: Show $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

6. *Hint*: Consider an arbitrary element of C.

7. (a) False.
   (b) True.
   (c) True.

8. *Hint*: Show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

1.2 Relations and Graphs

Solutions:

1. (a) Symmetric.

   (b) Reflexive, symmetric.

   (c) Reflexive, transitive, antisymmetric.

   (d) Reflexive, transitive, antisymmetric.

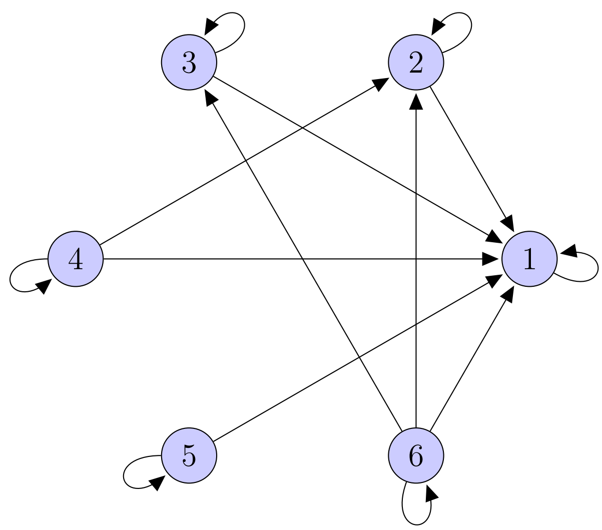
2. (a) The equivalence class of $z = 11$ is $[11]$ since 11 is prime. The other positive integers in $[11]$ are the positive integers whose largest prime divisor is 11.

   The number of equivalence classes is equal to the cardinality of all the prime numbers, which is infinite.

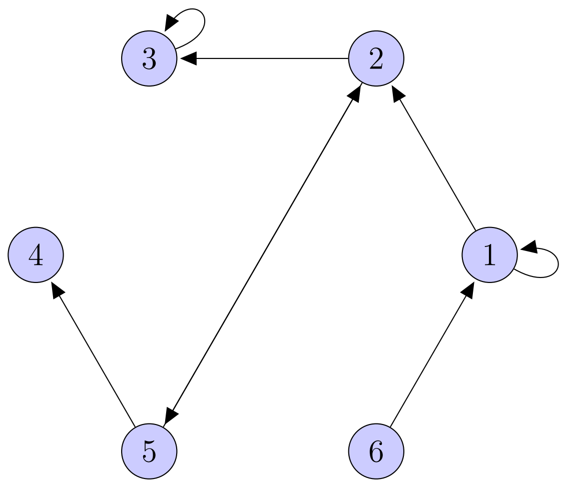
   (b) The equivalence class of $z = (2, 5)$ is $[29]$. This equivalence class includes all ordered pairs of real numbers such that $x_1^2 + x_2^2 = 29$.

   There are infinitely many equivalence classes of R, one for each positive real number that can be written as the sum of the squares of two real numbers.

3. (a)

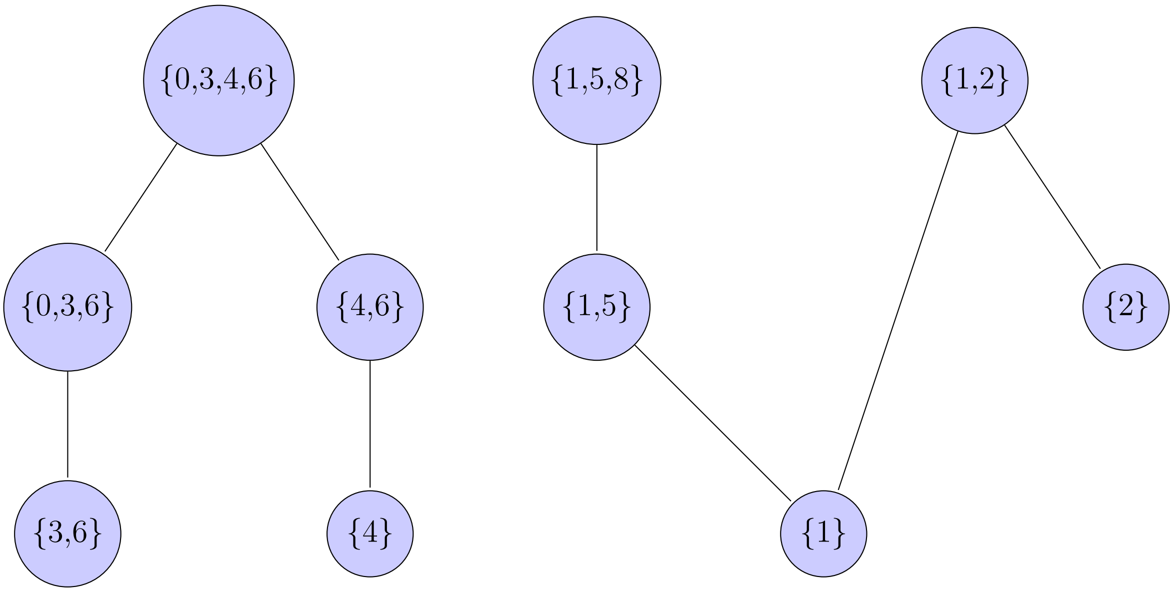


(b)



4. $R = \{(1,1),(3,3),(6,6),(1,3),(3,1),(1,6),(6,1),(3,6),(6,3),(2,2),(5,5),(2,5),(5,2),(4,4)\}$.

5.



6. Not a partial order.

7. (a) $2^{\frac{k(k+1)}{2}}$.

   (b) $2^k \cdot 3^{\frac{k(k-1)}{2}}$.

8. Yes antisymmetric.

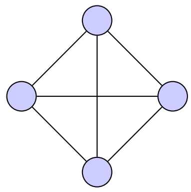
2 Graph Theory

2.1 Graphing Preliminaries

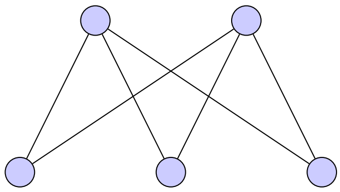
2.2 Definitions and Basic Properties

Solutions:

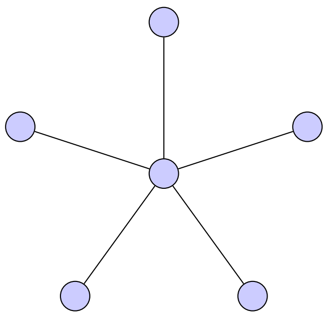
1. (a) 6 edges.



   (b) 6 edges.



   (c) 5 edges.



2. (a) $\frac{(n-1)(n)}{2}$ edges.

(b) $m \cdot n$ edges.

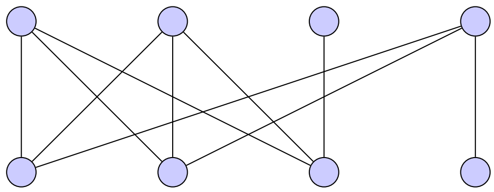
3. 16 edges.

4. (a) Does not exist.

   (b) Does not exist.

   (c) Does not exist.

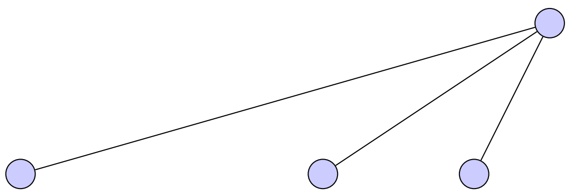
   (d) One possible graph:



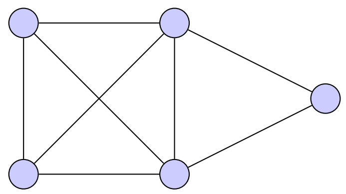
5. No.

6. (a) No.

   (b) One possible induced subgraph:



7. One possible graph:



8. Three.

9. Four.

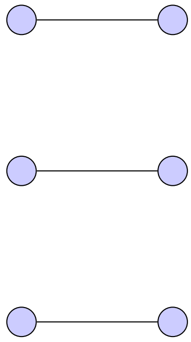
10. *Hint*: Euler's Theorem.

11. No.
    Hint: Cases.

12. Yes. This answer does not change if we require non-empty edges sets.

13. $d \in \{1, 2, 3, 5\}$.

14. (a)



(b) Does not exist.

(c) Does not exist.

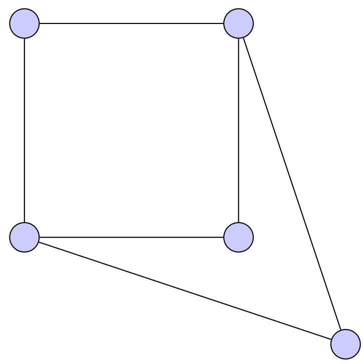
15. $2^{\frac{n(n-1)}{2}}$.

16. 23 vertices.

17. *Hint*: Euler's Theorem.

18. $n-1$.

19. One possible graph:



20. Impossible.

21. Yes.

22. *Hint*: Consider all possible degrees of an arbitrary vertex.

23. *Hint*: Proof by contradiction.

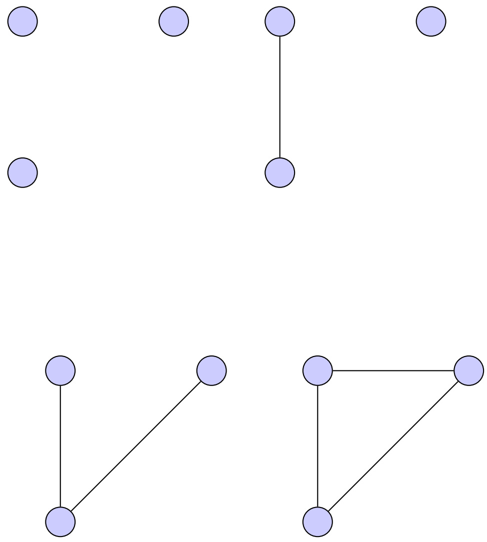
2.3 Isomorphisms

Solutions:

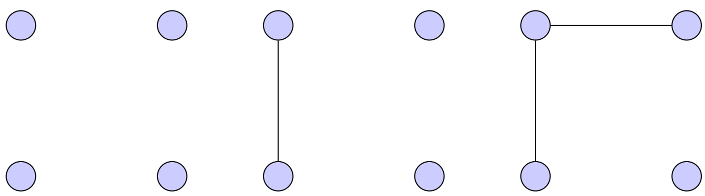
1. Two isomorphic graphs have the same structure.

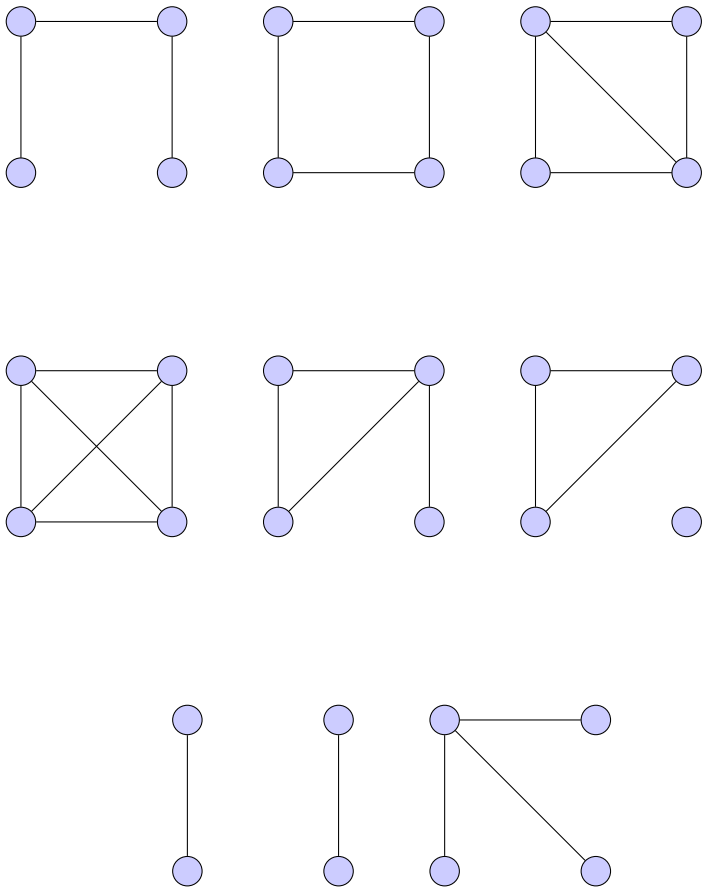
2. The converse of this statement: "If two graphs G_1 and G_2 have the same number of vertices, same number of edges, and the same degree sequence, then they are isomorphic". It is false.

3. (a)

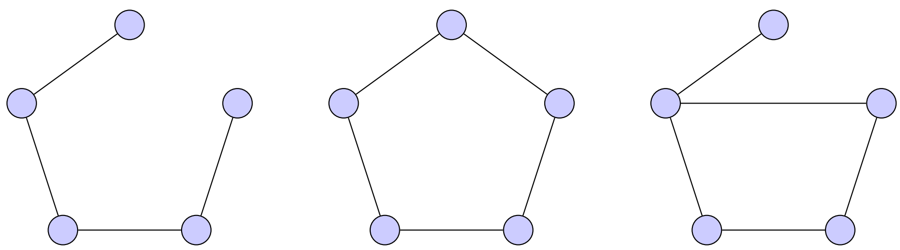


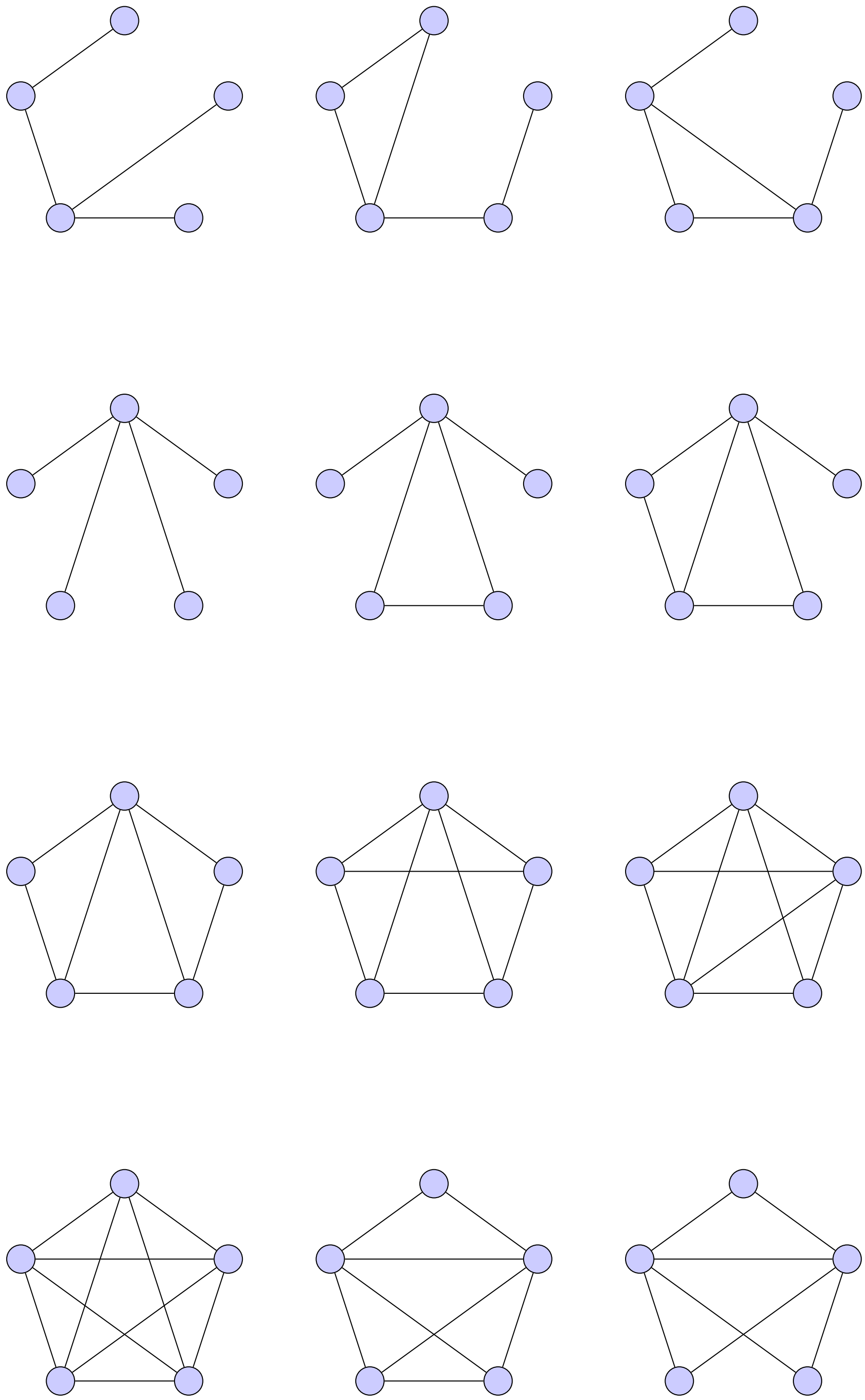
(b)

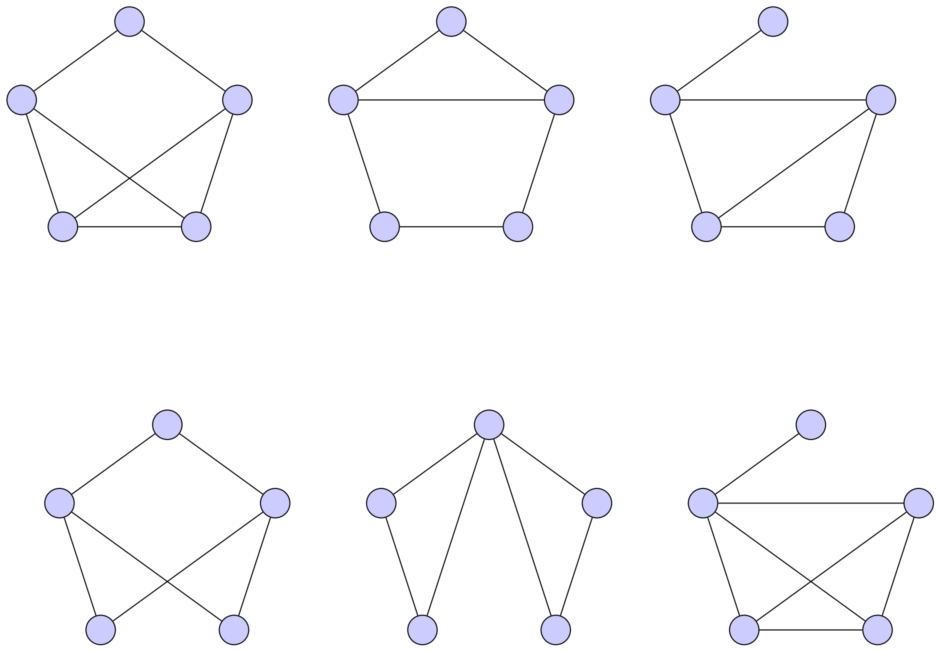




(c)







4. *Hint*: Recall the definition of a complete graph.

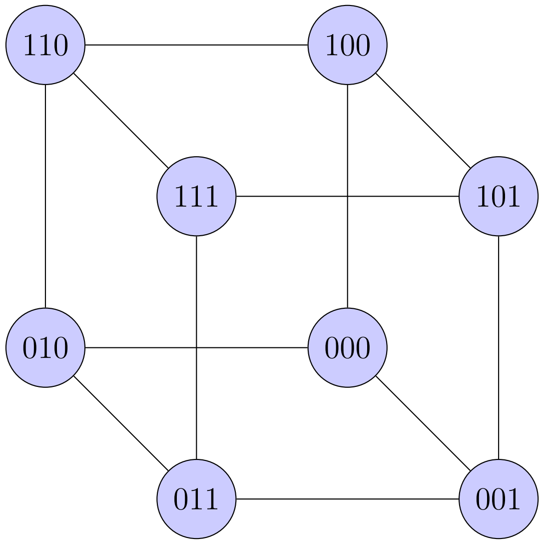
5. *Hint*: Use the formal definition of an isomorphism.

6. One possible bijection:

$$\begin{aligned} a &\longrightarrow m \\ b &\longrightarrow n \\ c &\longrightarrow p \\ d &\longrightarrow q \\ e &\longrightarrow r \end{aligned}$$

$$f \longrightarrow s$$
$$g \longrightarrow t$$
$$h \longrightarrow u$$

7.



8. (a) False.
   (b) True.
   (c) False.
   (d) True.
   (e) True.
   (f) False.
   (g) True.
   (h) True.

9. (a) Non-isomorphic.
   (b) Non-isomorphic.
   (c) Isomorphic.

(d) Isomorphic.

(e) Non-isomorphic.

10. *Hint*: There are exactly ten self-complementary graphs of order 8.

2.4 Eulerian Circuits

Solutions:

1. An Eulerian circuit is a trail that uses every edge exactly once and ends where it began. A Eulerian trail is a trail that goes through every edge, but does not necessarily end where it began.

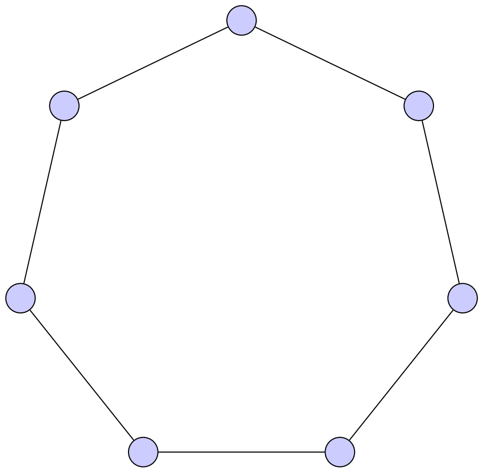
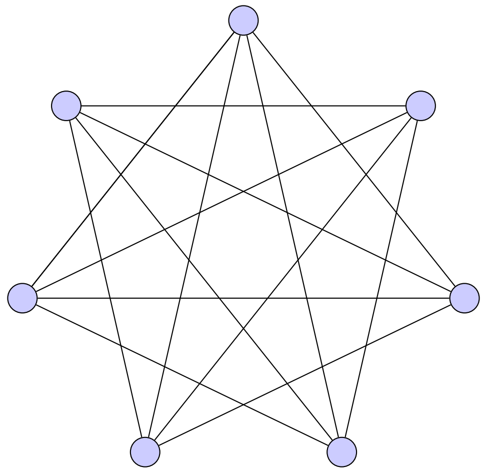
2. (a) No Eulerian circuit nor Eulerian trail.
   (b) No Eulerian circuit exists but there exists an Eulerian trail.
   (c) Both Eulerian circuit and Eulerian trail.
   (d) No Eulerian circuit nor Eulerian trail.
   (e) No Eulerian circuit nor Eulerian trail.
   (f) No Eulerian circuit nor Eulerian trail.

3. Not Eulerian.

4. *Hint*: Recall the Königsberg Bridge Problem: The city of Königsberg, Prussia, was set on both sides of a river and included two large islands, all connected by seven bridges. Is there a way to walk through the city crossing every bridge exactly once while finishing where you started?

5. *Hint*: By definition, every path is also a walk.

6. One possible graph:

7. (a) False.
   (b) True.
   (c) False.
   (d) True.

8. m, n both non-zero, even integers.

9. No.

10. (a) n odd.
    (b) For all odd values of n there will be a *closed* Eulerian trail (i.e. an Eulerian circuit). The only *open* Eulerian trail occurs when $n = 2$.

11. True.

Hint: Consider an arbitrary circuit.

12. *Hint*: Show the three properties of an equivalence relation hold.

13. *Hint*: Consider two arbitrary vertices in opposite components of a disconnected G.

14. *Hint*: Proof by contradiction.

15. $|E(G)| \geq n - 1$.

16. *Hint*: In any circuit, there exist two trails between any given pair of vertices.

2.5 Hamiltonian Cycles

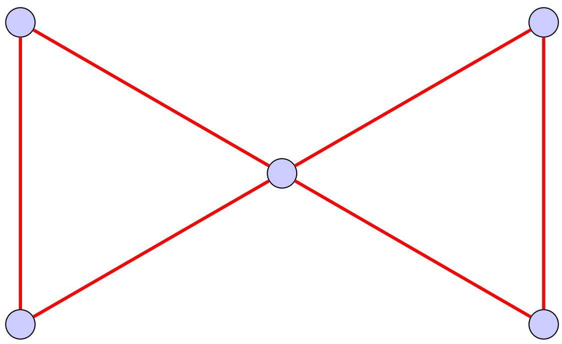
Solutions:

1. (a) In a Eulerian circuit it is possible to pass through some vertices multiple times while that is not possible in a Hamiltonian cycle. Also, a Hamiltonian cycle may not visit every edge while that is a requirement of a Eulerian circuit.

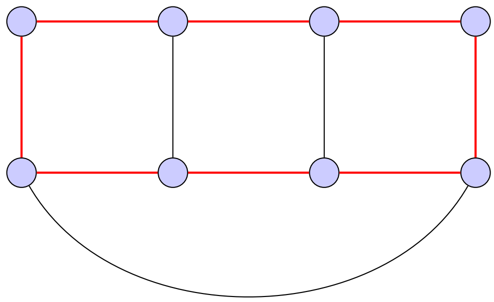
   (b) A Eulerian trail may visit the same vertex multiple times while a Hamiltonian path will not. A Hamiltonian path may not visit every edge in the graph, while that is a requirement of Eulerian trail.

2. (a) Not Hamiltonian.

   (b) The Hamiltonian cycle is highlighted in red.



   (c) One such cycle is highlighted in red.

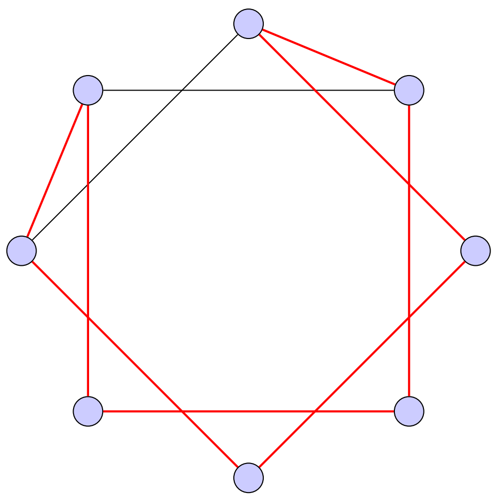


(d) Not Hamiltonian.

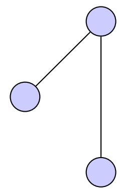
(e) Not Hamiltonian.

(f) Not Hamiltonian.

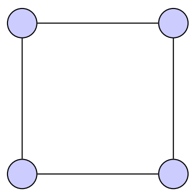
(g) Yes, a Hamiltonian cycle is highlighted in red:



3. The following graph is a sufficient counterexample:



4. Yes, many Hamiltonian and Eulerian graphs exist. A simple example:



5. *Hint*: Let each person be represented by a vertex and each friendship by an edge.

6. *Hint*: Show that G satisfies Ore's Theorem.

7. (a) False.
   (b) True.
   (c) True.

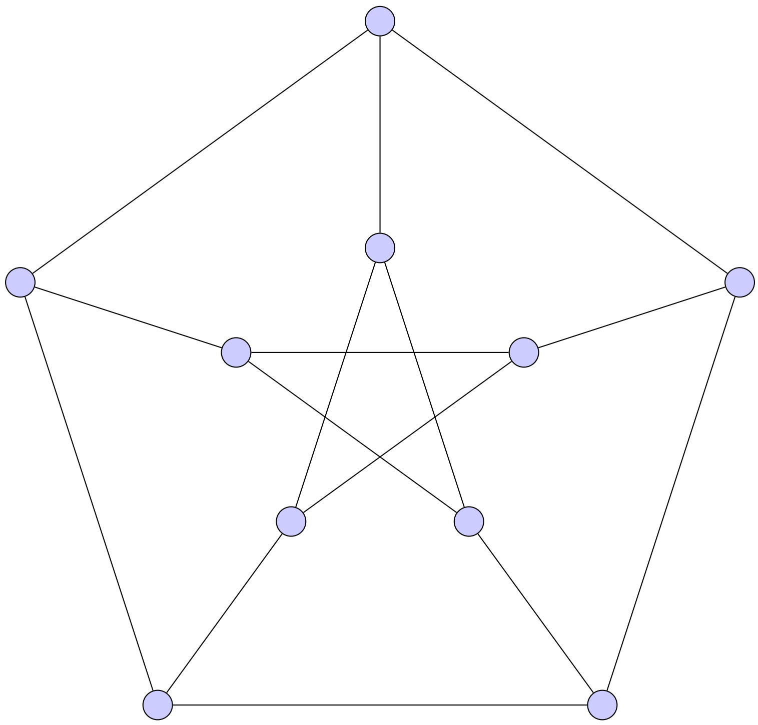
8. Yes Hamiltonian, not *necessarily* Eulerian.

9. $m = n$, with $m, n \geq 2$.

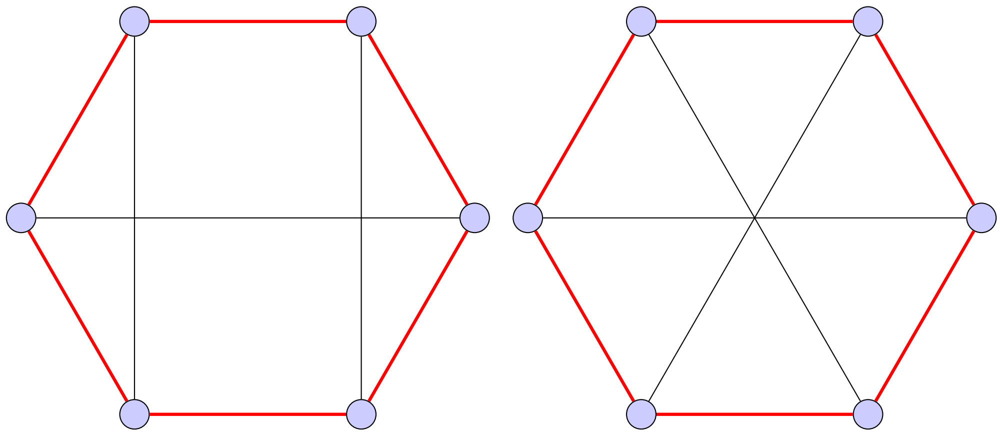
10. *Hint*: Consider an arbitrary cycle.

11. *Hint*: Proof by contradiction.

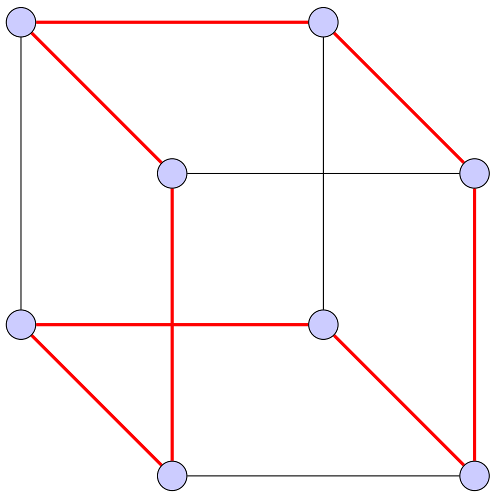
12. The Petersen graph is such a graph:



13. Here are the only such graphs, with Hamiltonian cycles highlighted in red.



14. Highlighted below is one of several Hamiltonian cycles in red:



2.6 Trees and Their Properties

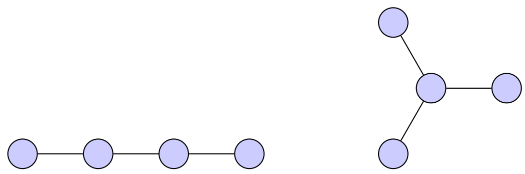
Solutions:

1. Infinitely many possible degree sequences exist. If you can draw a tree with your degree sequence then it is correct.

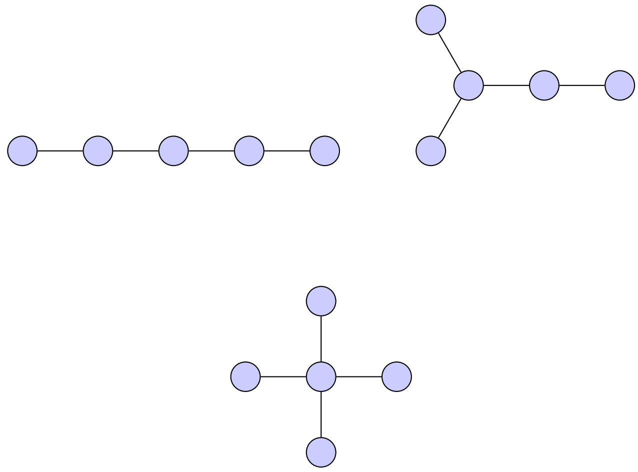
2. The following are equivalent:
   i) G is a tree.
   ii) G is a connected acyclic graph.
   iii) G is a connected graph with $n-1$ edges.
   iv) G is an acyclic graph with $n-1$ edges.

3. *Hint*: Consider the number of edges on a tree.

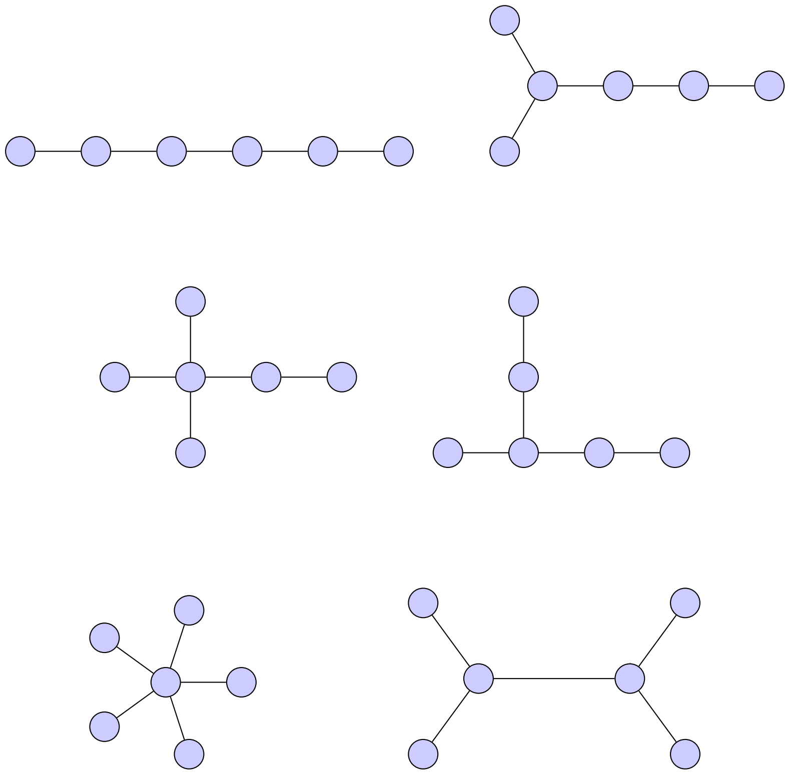
4. (a)



(b)



(c)



5. Yes.

6. False.

7. Nine vertices.

8. $x = 5$.

9. *Hint*: Induction on the number of vertices.

10. 102 vertices.

11. There are two vertices of degree 5.

12. *Hint*: Use the alternate definitions of a tree.

    The components of this new graph are trees.

13. *Hint*: Consider a property shared by trees and bipartite graphs.

14. *Hint*: Euler's Theorem.

15. A tree, T, is a complete bipartite graph if and only if $T = K_{1,n}$ for some positive integer n.

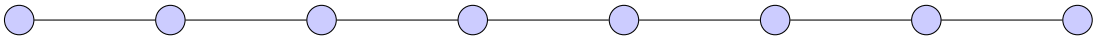
16. $|E(G)| = n - c$.

17. There are many possible graphs satisfying these properties. If your graph is disconnected or has a cycle then it is not a tree.

18. (a) Does not exist.

    (b) Does not exist.

    (c) One such tree:



    (d) Does not exist.

19. (a) There is no such spanning tree.

    (b) The subgraph induced by vertices b, e, f, g is one such induced 4-cycle.

2.7 Planar Graphs

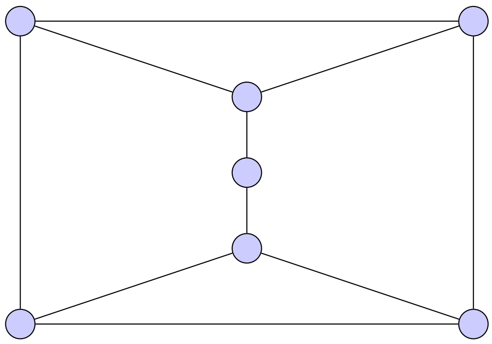
Solutions:

1. A graph is 'planar' if it can be drawn in the plane so that no edges cross.

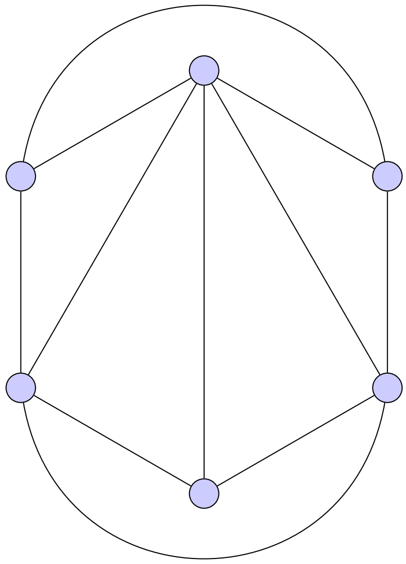
2. Two graphs are homeomorphic if one is a 'subdivision' of the other.

3. (a) This graph is nonplanar as it is homeomorphic to $K_{3,3}$.

   (b) This graph is planar.



   (c) This graph is planar.



(d) This graph is nonplanar as it is homeomorphic to K_5.

(e) This graph is nonplanar as it is homeomorphic to $K_{3,3}$.

4. *Hint* Induction on the order of the tree.

5. *Hint*: Apply Euler's Planar Graph Theorem.

6. *Hint*: Consider what happens if a region was bounded by more than three edges.

7. $n \in \{1, 2, 3, 4\}$.

8. Without loss of generality, if $m \leq 2, n \leq 3$ then $K_{m,n}$ is planar.

9. *Hint*: Draw P_2.

10. *Hint*: Use Euler's Theorem and Euler's Planar Graph Theorem.

11. *Hint*: Euler's Planar Graph Theorem.

12. (a) False.
    (b) True.
    (c) True.
    (d) True.

13. (a) Such a graph does exist and has exactly 8 regions.

    (b) Such a graph does exist and has exactly 9 edges.

    (c) There is no such planar graph.

    (d) There is no such planar graph.

14. Does not exist.

15. *Hint*: How many edges do these two regions share?

2.8 Colouring Graphs

Solutions:

1. Assigning every vertex of a graph a colour such that no adjacent vertices have the same colour.

2. The graph is edge-less/empty.

3. (a) $\chi(K_n) = n$.
   (b) $\chi(K_{m,n}) = 2$.
   (c) $\chi(G) = 2$.
   (d) $\chi(G) = 2$.
   (e) $\chi(G) = 3$.
   (f) $\chi(G) = 2$.
   (g) $\chi(G) = 4$.
   (h) $\chi(G) = 2$.
   (i) $\chi(G) = 4$.

4. False.

5. (a) False.
   (b) False.
   (c) True.
   (d) False.
   (e) False.
   (f) True.

(g) False.

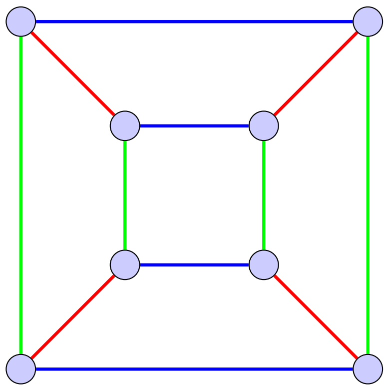
(h) False.

(i) False.

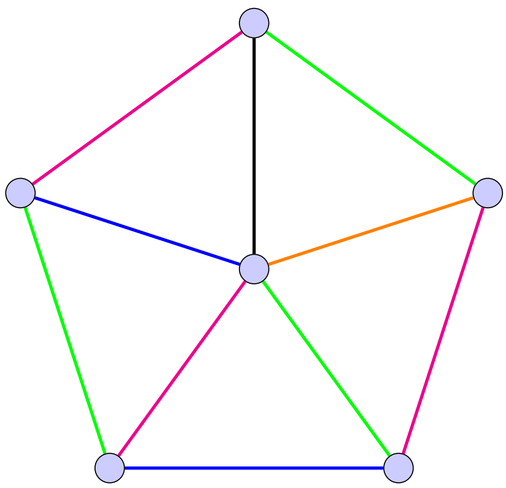
(j) True.

(k) False.

6. (a) Here is one such minimum edge colouring:



(b) Here is one such minimum edge colouring:



7. (a) Every animal represents a vertex. Two vertices are adjacent (i.e. there is an edge) if these animals *cannot* live together peacefully. The vertices assigned the same colour represent the animals that can live in the same enclosure. The zoo is attempting to find the chromatic number of such a graph.

   (b) Let each course be a vertex, with two vertices adjacent if a student indicates that they would like to enroll in both courses. Vertices assigned the same colour represent courses that can run at the same time. Any colouring of this graph will give the department such a schedule, however the most efficient schedule would be indicated by the chromatic number of the graph.

8. *Hint*: Cases and proof by contradiction.

9. *Hint*: Induction on the order of the graph.

10. True.

11. *Hint*: Consider three distinct cases.

12. There are no adjacent vertices within a set of vertices of the same colour.

3 Counting: Fundamental Topics

3.1 Basic Counting Principles

3.2 The Rules of Sum and Product

Solutions:

1. (a) The sum rule for multiple events is for events that are *independent* (i.e. events/situations that cannot occur at the same time). If one event can occur m ways, and the other event can occur in n ways, where both events are independent, then the two events together can occur in $m + n$ ways.

   (b) The product rule for multiple events is for events that are happening in sequence of one another, thus are *not* independent. For example, if one event can occur in m ways and then another event follows and can occur in n ways, then the sequence of these two events can occur in mn ways.

2. The product and sum rule are used together when a set of sequences of events are occurring independently.

3. (a) *Rule of product.*

   36^6.

   (b) *Rule of product.*

   35 880 000.

   (c) *Rule of product and rule of sum.*

   25 309 440.

(d) *Rule of product and rule of sum.*

1 021 798 336.

(e) *Rule of product and sum.*

6 718 464.

4. *Rule of sum.*

   58.

5. *Rule of sum and product.*

   231.

6. *Rule of product.*

   36.

7. *Rule of product.*

   22.

8. (a) *Rule of product.*

   $5 \cdot 9 \cdot 10^4$.

(b) *Rule of product.*

$5 \cdot 9^4$.

(c) *Rule of product.*

62 784.

9. *Rule of product.*

1 536.

10. *Rule of product.*

$5 \cdot 9 \cdot 10$.

11. *Rule of product.*

456 976.

12. *Rule of product.*

(a) $n^{m-1} \cdot 1$.

(b) $n^{m-2} \cdot n - 1$.

(c) $3 \cdot n^{m-1}$.

(d) $n \cdot n - 1^{m-1}$.

(e) $n^{m-1} \cdot n - 1$.

13. *Rule of product.*

    3^5.

14. *Rule of product.*

    2^{25}.

15. *Rule of product and sum.*

    1 550.

16. *Rule of product.*

    60.

17. *Rule of product.*

    Lock 1 has a higher number of possible combinations and hence is the lock Jamie should purchase.

18. *Rule of sum and product.*

    18 278.

3.3 Permutations

Solutions:

1. A permutation is a linearly ordered arrangement of distinct objects.

2. If objects could be repeated, two identical arrangements would be counted as different.

3. (a) 120.
   (b) 60.
   (c) 5^{10}.

4. 20.

5. (a) 5 040.
   (b) 1 440.

6. (a) 6 375 600.
   (b) 2 772.

7. 30 240.

8. 1 440.

9. 27 907 200.

10. 100.

11. (a) 3 315 312 000.
    (b) 892 584 000.
    (c) 127 512 000.
    (d) 1 356 727 680.

12. $m!(n+1)!$.

13. Applying the definition,

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

14. The permutation function is one specific application of the product rule if you are trying to determine the number of possible arrangements of k out of n distinct objects ($k \leq n$). If this is not the specific case, count using the Rule of Product.

15. Solve your own problem to verify this.

16. *Hint*: Use the definition of the permutation function.

17. 5 184.

18. 34 650.

19. $|A| = m \leq n = |B|$, otherwise no one-to-one functions can exist between the sets.

   (a) 0.
   (b) $P(n-1, m-1)$.
   (c) $\frac{3(n-1)!}{(n-m)!}$.
   (d) $P(n, m)$.
   (e) $P(n, m)$.

20. (a) 151 200.
   (b) 70 560.
   (c) 10 080.
   (d) 3 600.

3.4 Combinations and the Binomial Theorem

Solutions:

1. The main difference between a permutation and a combination is that for a permutation the order in which the elements are selected matters while for a combination the order does not matter. Both a combination and a permutation count the ways in which an event can occur.

   The formula for permutation, $P(n,r)$, is $\frac{n!}{(n-r)!}$ while the formula for combination, $C(n,r)$ is $\frac{n!}{(n-r)!r!}$. Clearly, $C(n,r) = \frac{P(n,r)}{r!}$ which highlights that order is irrelevant in a combination.

2. There are $\binom{69}{5}\binom{26}{1} = 292\ 201\ 338$ possible tickets. Thus, the cost of buying all the tickets is higher than the prize money and so it is not worth it to buy all the tickets to ensure a win.

3. (a) $\binom{80}{21}$.
   (b) $\binom{40}{10}\binom{40}{11}$.
   (c) $\binom{78}{19}$.
   (d) $\binom{65}{10}\binom{15}{11}$.
   (e) $\binom{20}{4}\binom{15}{5}\binom{10}{4}\binom{20}{4}\binom{15}{4}$.
   (f) $\binom{75}{16}\binom{5}{5} + \binom{75}{17}\binom{5}{4} + \binom{75}{18}\binom{5}{3}$.

4. $n = 91$.

5. $\binom{13}{3}$.

6. (a) $\binom{12}{6}$.

(b) $\binom{10}{5}\binom{2}{1}$.

7. $\binom{10}{3} - \binom{8}{1} = \binom{9}{2} + \binom{9}{2} + \binom{8}{3}$.

8. $\binom{15}{7} - \binom{12}{5} = \binom{2}{1}\binom{13}{6} + \binom{13}{7} + \binom{12}{4}$.

9. (a) $\binom{20}{8}$.
   (b) $\binom{10}{4}\binom{10}{4}$.
   (c) $\binom{20}{8} - \binom{18}{6} = \binom{18}{8} + \binom{2}{1}\binom{18}{7}$.
   (d) $\binom{20}{8}\binom{8}{1} = \binom{20}{1}\binom{19}{7}$.

10. **Algebraic:**

$$\begin{aligned} m\binom{n}{m} &= n\binom{n-1}{m-1} \\ \frac{m(n!)}{(n-m)!m!} &= \frac{n(n-1)!}{(n-1-(m-1))!(m-1)!} \\ \frac{n!}{(n-m)!(m-1)!} &= \frac{n!}{(n-m)!(m-1)!} \end{aligned}$$

Clearly from above, the two side are algebraically equivalent.

Combinatorial: Consider a group of n people who all apply to be on a committee of m people that requires a leader. There are two possible ways we can form the committee. We can either first choose from the larger group of n people our committee of m individuals, and then within that committee chose a leader, m possibilities. This is $m\binom{n}{m}$.

Alternatively, we can pick from the leader from the larger group of n people first, and then from the remaining $n-1$ select the remaining $m-1$ non-leader committee members. This is $n\binom{n-1}{m-1}$.

Since we counted the same scenario in two different ways, these expressions are equivalent.

11. We can look at this problem as counting the number of subsets given a set. We know that 2^n counts the total number of subsets from a set of cardinality n. This is because for each element in the set we are given two 'choices': whether or not to include it in the set. Since there are n elements in a set, we can use the product rule to see there are a total of 2^n possible subsets.

    Alternatively, we know that $\binom{n}{k}$ is the total number of subsets of size k from a set of size n. And so, the left side is the sum of all possible subsets of n of size 0 to n. This is all possible sizes of subsets and so this is all possible subsets of a set of size n.

    Since we counted the same scenario in two different ways, these expressions are equivalent.

12. Suppose a teacher in a classroom of n students is looking to take k students to a special conference. We know that there are $\binom{n}{k}$ to select these students, which is the left side of our equation.

    We can also look at this problem in two cases. Suppose there is a student Chan Ming in the class. We can look at two cases relating to Chan Ming either attending the conference or not. If Chan Ming does not attend the conference there are $\binom{n-1}{k}$ ways of picking students to go to the conference. If Chan Ming does attend the conference there are $\binom{n-1}{k-1}$ ways of selecting the remaining students to attend the conference. Thus, by the Rule of Sum, the total number of possible groups of students to attend the conference is $\binom{n-1}{k} + \binom{n-1}{k-1}$.

    Since we counted the same scenario in two different ways, these expressions are equivalent.

13. (a) $\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$.

 (b) $\sum_{k=0}^{6}(3)^k(-x)^{6-k}\binom{6}{k} = x^6 - 18x^5 + 135x^4 - 549x^3 + 1215x^2 - 1458x + 729$.

 (c) $\sum_{k=0}^{7}(2x)^k(-3y)^{7-k}\binom{7}{k} = 128x^7 - 1344x^6y + 6048x^5y^2 - 15120x^4y^3 + 22680x^3y^4 - 20412x^2y^5 + 10206xy^6 - 2187y^7$.

 (d) $\sum_{k=0}^{15}(4x)^k(7y)^{n-k}\binom{15}{k}$.

14. (a) $\binom{13}{9}$.
 (b) $2^9\binom{13}{9}$.
 (c) $\binom{13}{9} \cdot 4^9 \cdot (-3)^4$.

15. (a) $2^7 \cdot 3^4 \cdot \binom{11}{7}$.
 (b) $2^7 \cdot 5^2 \cdot \binom{9}{7}$.
 (c) 3^5.
 (d) $(-2)^3 \cdot (2)^9 \cdot \binom{12}{3}$.
 (e) $2 \cdot 4^6 \cdot 7$.

16. (a) $3^n - 1 - 2n$.
 (b) 0.

17. *Hint*: Cases for the parity of k.

18. (a) 7^n.

(b) 2^{3n}.

(c) $(-1)^n$.

(d) $6 \cdot 2^n$.

(e) 0.

19. When evaluating a polynomial with more than two terms to some integer power the multinomial theorem is used to determine the coefficients of the terms. The binomial theorem is a version of the multinomial theorem that can be used for binomials.

20. (a) 12.

    (b) 816 480.

    (c) There is no term with xyz.

3.5 Combinations with Repetitions

Solutions:

1. i) The number of ways r identical elements can be distrbuted into n distinct containers.
   ii) The number of non-negative integer solutions to:
   $$x_1 + x_2 + ... + x_n = r.$$

2. We are selecting r elements, with possible repetition, from a set of n distinct objects.

3. 330.

4. 41 120 525.

5. (a) $\binom{40}{12}$.
   (b) $\binom{51}{12}$.

6. (a) $\binom{20+m-1}{20}$.
   (b) m^{20}.

7. 1 680.

8. There are many possible correct answers, one example is, in how many can you distribute ten treats between three dogs?

9. 42 504.

10. 56.

11. 2 925.

12. 26 334.

13. 126.

14. $n = 7$.

15. $n = 7$.

16. **Algebraically**: Expanding using the definition of the combination function, we know,

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!},$$

and that,

$$\binom{n+r-1}{n-1} = \frac{(n+r-1)!}{(n-1)!r!}.$$

It is easy to see that these expansions are equal since multiplication is commutative, so we're done.

Combinatorial proof: We can look at this problem as placing r balls into n boxes.
If we wish to do this, we can line up the r balls and place $n-1$ dividers between

them. The balls between either the beginning/end and a divider or two dividers represents the number of balls in a box. Thus, there are $n-1+r$ total positions, where each position is either filled with a ball or a divider.
From these positions we can either choose where to first place the dividers and then have the balls fill the remaining positions, $\binom{n+r-1}{n-1}$, or we can choose where to place the balls first and have the dividers fill the remaining positions, $\binom{n+r-1}{r}$. The two options are equivalent and are equal to the right and left sides of the equation, respectively.

17. (a) $P(n, r)$

    (b) $C(n, r)$

    (c) n^r

    (d) $C(n+r-1, r) = C(n+r-1, n-1)$

18. 136.

19. Determine the number of integer solutions to:

    (a)

    $$x_1 + x_2 + x_3 = 7,$$

    where $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$ and $0 \leq x_3 \leq 2$. We have x_1 representing the red marbles, x_2 the blue and x_3 the green marbles.

    (b)

    $$x_1 + x_2 + x_3 + x_4 + x_5 = 30,$$

    where $x_i \geq 0$ for $i = 1, 2, 3, 4, 5$. Each x_i represents how many poker chips each individual has. (We require each x_i to be positive since we are modelling a situation where having negative poker chips is nonsensical.)

(c)

$$x_1 + x_2 + x_3 + x_4 = 12,$$

where $x_i \geq 2$ for $i = 1, 2, 3, 4$. Each x_i represents the number of apples of each variety that have been selected.

(d)

$$x_1 + x_2 + x_3 + x_4 = 15,$$

where $x_i \geq 0$ for $i = 1, 2, 3, 4$ with $x_1 = x_2$. Each x_i represents the amount of markers in each box.

20. $\binom{n-m+x-1}{x} \cdot \binom{n}{m}$.

21. $\binom{m+s-1}{s} \cdot \binom{n+r-s-1}{r-s}$.

3.6 The Pigeonhole Principle

Solutions:

1. If there are k pigeons that are flying into n pigeonholes where $n < k$, then there must be at least one box with at least two pigeons.

2. A function from one finite set to a smaller finite set cannot be one-to-one. There will be at least two elements from the domain that map to the same image in the co-domain/range. The pigeons represent the domain and the pigeonholes the co-domain. The function is the assignment of pigeons to pigeonholes.

3. There is nothing specific that can be said, unless $m > n$. If $m > n$ we know that at least one pigeonhole will be empty. Beyond that, there are many possible arrangements of the pigeons.

4. First notice if $k = 1$, this is precisely the pigeonhole principle.

   Consider when $k > 1$. Suppose for contradiction that each pigeonhole houses at *most* k pigeons. Then there are, at most, $k \times n = kn$ pigeons, which is a contradiction as there are $kn+1$. Thus, at least one pigeonhole hosts $k+1$ pigeons.

5. (a) There are 366 possible birthdays, including February 29. As $367 > 365$, there will be at least two people who have the same birthday.

   The pigeons are the people and the pigeonholes are the birthdays.

   (b) Every integer can be written in the form $28k + m$, where $k \in \mathbb{Z}$ and $m \in \{0, 1, ..., 27\}$. So, m represents the remainder of the integer when divided by 28. There are 28 possible values of m. Therefore, there must be at least two integers in the set of 29 integers with the same remainder.

The pigeons are the different integers and the pigeonholes represent the possible remainders upon division by 28.

(c) You must take out at least 10 shoes before you are guaranteed to obtain a pair. Any less and it is possible that each shoe is from a different pair.

The pigeons are the shows and the pigeonholes are the pairs of shoes.

(d) To determine this, we must first count the number of distinct three letter words. For each position in the word there are 26 possible letters. Therefore, in total there are $26^3 = 17\,576$ distinct three letter words. This means that it is possible, but not definitive, that all words on this list are distinct as there are more three-letter words possible than there are words on the list.

The pigeons are the number of three-letter words and the pigeonholes are the words on the list.

6. *Hint*: Prove the statement about subsets of size 6, the result will follow for all larger subsets.

7. (a) Any even positive integer can be written as $x = 2^k y$ (essentially just factoring out the two's), where $k \in \mathbb{N}$, and y is odd. There are exactly 1000 odd numbers in $A = \{1, ..., 2000\}$.

Let us define a pigeonhole for each odd integer $y \in A$ as:

$$PH_y = \{x \in A : x = 2^k y, \text{ where } y \text{ is odd and } k \geq 0\}.$$

This gives us our 1000 pigeonholes.

Select any 1001 numbers from A, these are the pigeons. Then, by the pigeonhole principle, there exists a pigeonhole, PH_y, that contains two selected numbers: a and b. Say $a = 2^k y$ and $b = 2^p y$ for some distinct, nonnegative integers k and p. If $k > p$, then b divides a and if $k < p$, then a divides b.

(b) We can partition the numbers into sets of size two, where the second digit is one less than the first: $\{1, 2\}, \{3, 4\}, ..., \{1997, 1998\}, \{1999, 2000\}$. Then, there are exactly 1000 of these disjoint subsets, which represent our 'pigeonholes'.

Choose any 1001 integers and let them represent our 'pigeons'. Then by PHP, we will have two integers from the same disjoint subset. Hence, two integers are relatively prime.

8. First determine the number of possible, distinct initials. There are 26 options for each one's first and last initial. Therefore, there are $26^2 = 676$ different possible initials.

   Let the attendees represent the 'pigeons' and the possible initials represent the 'pigeonholes'. To guarantee there are at least two attendees with the same initials, it follows from the Pigeonhole Principle that we need more than 676 attendees. So, we require at least 677 attendees.

9. Notice that Brynn spent $6 \cdot 7 = 42$ days sending out scholarship applications.

   For $1 \leq i \leq 42$, let x_i represent the number of scholarships Brynn has sent out in total as of day i. Since Brynn sends out at least one application per day and no more than 60 total, we know that $1 \leq x_1 < x_2 < ... < x_{42} < 60$.

   Now adding 23 to every term of the inequality we obtain:

$$1 + 23 = 24 \leq x_1 + 23, x_2 + 23, ..., x_{42} + 23 < 60 + 23 = 83$$

Note: Recognizing that you need to do this is the heart of the proof. We add 23 because we are trying to prove that there are 23 consecutive days where scholarship applications were sent out, which allows us to conclude this by the PHP.

Now we have 84 distinct numbers, $\{x_1, x_2, ..., x_{42}, x_1 + 23, x_2 + 23, ..., x_{42} + 23\}$. Let these numbers represent out 'pigeons'. These 84 numbers must all lie between 1 and 83, where the range of integers from 1 to 83 represent out pigeonholes. Thus by the PHP there exists an $x_i = x_j + 23$ for some $i > j \in \{1, ..., 42\}$. This means that from the beginning of day $j + 1$ to the end of day i, Brynn applied for 23 scholarships.

10. It suffices to prove the result for subsets of exactly size three, since that will imply the result for subsets of size larger than three.

    The only way for the sum of two integers to be even is if both of the integers have the same parity, that is both are even or both are odd. Any given integer can be classified as either even or odd, hence any subset of 3 integers will contain at least two with the same parity by the PHP. Thus, there are two integers in the subset with an even sum.

11. There are 12 pigeonholes (computers) and 42 pigeons. In this problem there is a restriction that no pigeonhole can hold more than 6 pigeons.

    We wish to show that there are five computers which are used by three or more people.

    Let us assume for a contradiction that this is not true. This would mean that 8 computers are used by at most 2 people. This would mean that these 8 computers are used by at most 16 people all together.

    There are 42 people who use a computer at the library and so that means the remaining 26 people use 4 computers.

This means that there are 26 pigeons and 5 pigeon holes where the maximum capacity of each pigeonhole is 6. This however gives that the maximum capacity for the remaining computers is 24, which is a contradiction.

So, at least 5 computers are to be used by three or more people.

12. Let us assume that if one person speaks to another, the person will respond. That is, assume speaking to someone is a reflective relation.

    If there are n people at the party, each person can speak to between 0 and $n-1$ people, as no person can speak to themselves and speaking to someone is reflective.

    If a person spoke to $n-1$ people, then it is impossible for any person to have spoken to 0 people. In this case every person spoke to between 1 and $n-1$ people. That means there are $n-1$ potential number of people a person could have spoken to.

    If a person at the party spoke to 0 people, then it is impossible for someone to have spoken to everyone. In this case every person will have spoken to between 0 and $n-2$ people. That means there are $n-1$ potential number of people a person could have spoken to.

    In both of the above cases, there are $n-1$ potential amounts of people a person could have spoken to but n people. Thus, by the PHP two people will have spoken to the same amount of people at the party.

13. For 12 to divide the difference of two numbers, they must have the same remainder upon division by 12. Observe that $12k+m-(12j+m)=12(k-j)$,

where $k, j, m \in \mathbb{Z}$ and m represents the remainder of the arbitrary integer when divided by 12.

Certainly the only possible remainders are $\{0, ..., 11\}$, of which there are 12 possibilities. Thus by the PHP, as 12 integers have been selected at least two must have the same remainder when divided by 12. Thus, their difference is divisible by 12, as desired.

14. Notice that a rectangle with width of 3 metres and height of 4 metres has a diagonal length of 5 metres. If we divide the field into rectangles of this size, we are able to split the field into 30 rectangles. Let the cows represent 'pigeons' and the rectangles represent 'pigeonholes'. Then, by the PHP, at least two cows must be in the same rectangle. The farthest two cows are apart in a rectangle in 5 metres and so the result follows.

15. In A, there are exactly 40 integers divisible by 5. Therefore one must select $200 - 40 + 1 = 161$ integers to guarantee that at least one of them is divisible by 5.

16. There are exactly 30 odd numbers bin X, and 61 numbers to choose from. Therefore at least $61 - 30 + 1 = 32$ numbers must be selected to guarantee that at least one is odd.

4 Inclusion and Exclusion

4.1 The Principle of Inclusion-Exclusion

Solutions:

1. The Principle of Inclusion-Exclusion is a counting method that ensures every possible event is counted, while also taking into account events that can co-occur. This method is a way of ensuring events are not counted twice or "double counted".

2. False.

3. (a) 16.
   (b) 27.
   (c) 20.

4. 300.

5. (a) 46.
   (b) 35.

6. (a) There is no correct answer - the question is unanswerable because the situation described is impossible.
   (b) 0.
   (c) 135.

7. $11! - \sum_{i=1}^{6}(-2)^i\binom{6}{i}(11-i)!$

8. (a) 1 714.
   (b) 214.
   (c) 1400.
   (d) 63.

9. $10^9 - 3(9^9) + 3(8^9) - 7^9$.

10. $10! - 2 \cdot 9! - 2 \cdot 9! + 2^2 \cdot 8!$

11. $\frac{26!}{16!} - \frac{6(21!)}{16!} - \frac{8(23!)}{16!} - \frac{7(22!)}{16!} + \frac{4(19!)}{16!} + \frac{3(18!)}{16!} + \frac{4 \cdot 3 \cdot (19!)}{16!} - 1$.

12. $\frac{26!}{14!} - \frac{10 \cdot 23!}{14!} - \frac{9 \cdot 22!}{14!} - \frac{8 \cdot 21!}{14!} + \frac{6 \cdot 7 \cdot 19!}{14!} + \frac{4 \cdot 5 \cdot 17!}{14!}$.

13. $\frac{11!}{4!4!2!} - \frac{8!}{4!2!} - \frac{10!}{4!4!} - \frac{8!}{4!2!} + \frac{7!}{4!} + \frac{5!}{2!} + \frac{7!}{4!} - 4!$.

14. (a) 3 276.
    (b) 348.
    (c) 1 509.

15. $\binom{30}{5} - \binom{3}{1} \cdot \binom{23}{17} + \binom{3}{2} \cdot \binom{16}{10} - \binom{9}{3}$.

16. *Hint*: Consider an arbitrary $x \in S$ and see what happens when a certain number of the conditions are satisfied by x.

4.2 Derangements: Nothing in its Right Place

Solutions:

1. Simply, a derangement is a permutation where no element appears is in its original position. Formally, a derangement is a function, f, on a set X such that for all $x \in X$, $f(x) \neq x$.

2. Derangements are one specific application of the PIE. In order to find the number of ways to arrange items such that nothing is in its original place, we could use the PIE to exclude all the cases where things *are* in their right place. (Remember that PIE is defined by satisfying *none* of the conditions.)

3. $d(26)$.

4. (a) $d(150)$.
   (b) $d(50) \cdot d(30)^2 \cdot d(40)$.

5. $5! \cdot d(5)$.

6. $d(10) + 5 \cdot d(9) + 10 \cdot d(8) + 10 \cdot d(7) + 5 \cdot d(6) + d(5)$.

7. $n = 10$.

8. (a) $d(8)$.
   (b) $8! - d(8)$.
   (c) $28 \cdot d(6)$.
   (d) This is impossible.

9. (a) $[d(12)]^2$.

   (b) $[d(6)]^2$.

   (c) $\sum_{k=0}^{12}(-1)^k[(12-k)!]^2\binom{12}{k}$.

   Hint: Use PIE instead of derangements.

10. We will count the number of permutations of the numbers $1, 2, 3, ..., n$, which is certainly $n!$. Alternatively, for every possible permutation we can consider how there are k elements that have been deranged, and hence $n-k$ elements in their original positions for $0 \leq k \leq n$. The $n-k$ fixed elements can be selected in $\binom{n}{n-k} = \binom{n}{k}$ ways, with $d(k)$ ways that the k remaining elements can be deranged. We sum these cases from $k = 0$ to $k = n$ to account for all possible permutations, and the proof is complete since we've counted the same situation in two different ways.

4.3 Onto Functions and Stirling Numbers of the Second Kind

Solutions:

Note: There are two equivalent formulas for Stirling numbers:

$$S(m,n) = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m$$

These formulas are equivalent as:

$$\begin{aligned}\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m &= \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m + (-1)^n\binom{n}{n}(n-n)^m \\ &= \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m + 0 \\ &= \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m\end{aligned}$$

Thus, you may use either in your solutions.

1. An onto function is a function $f : A \longrightarrow B$ where for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

2. The number of ways to distribute n different objects into m distinct containers where no container is left empty and $n \geq m$.

3. There are many possible examples, one example is the function $y = x$ where $x \in \mathbb{Z}$.

4. $|A| \geq |B|$.

5. No such surjective function.

6. $\sum_{k=0}^{8}(-1)^k\binom{9}{k}(9-k)^{13}$.

7. 2.

8. $\sum_{k=0}^{6}(-1)^k \binom{7}{7-k} (7-k)^{27}$.

9. 2 100.

   Hint: Two cases.

10. A Stirling number of the second kind, denoted $S(m,n)$, is the number of ways to distribute m distinct objects into n identical containers with no container left empty.

    The formula is: $S(m,n) = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k \binom{n}{n-k} (n-l)^m$. This is the formula for counting the number of onto functions from a set of size m to a set of size n divided by $n!$. The division by $n!$ is done to account for the identical "container".

11. $j! \cdot S(k,j)$.

12. (a) 5 103 000.

    (b) 1 020 600.

    Hint: Cases depending on if any other stuffed animals go into the first bin.

(c) 86 472.

Hint: Six cases.

13. We recall from number theory that any factor of 55 335 will be the product of some subset of the factors of 55 335. For example. $3 \cdot 5$ and $31 \cdot 3 \cdot 17$ are both factors of 55 335.

    (a) $S(5, 2) = 15$.

    (b) $\sum_{n=2}^{5} S(5, n) = 51$.

    Hint: Four cases.

14. *Hint*: Expand the sums.

5 Generating Functions

5.1 Introductory Examples

5.2 Definition and Examples: Calculating Techniques

Solutions:

1. (a) $$1 + 2x + 3x^2 + 4x^3 + ... + nx^{n+1} + ... = \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \frac{1}{(1-x)^2}$$

   (b) $$5 + 4x + 3x^2 = \sum_{k=0}^{2} \binom{5-k}{1} x^k.$$

   (c) $$1 - x + x^2 - x^3 + x^4 - x^5 + ... + (-1)^n x^n + ... = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1+x}.$$

   (d) $$\binom{10}{10} + \binom{11}{10} x + \binom{12}{10} x^2 + ... = \sum_{k=0}^{\infty} x^k \binom{10+k}{10} = \frac{1}{(1-x)^{11}}$$

   (e) $$\binom{10}{10} - \binom{11}{10} x + \binom{12}{10} x^2 - \binom{13}{10} ... = \sum_{k=0}^{\infty} (-x)^k \binom{10+k}{10} = \frac{1}{(1+x)^{11}}$$

   (f) $$1 + x^2 + x^4 + ... = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$$

   (g) $$1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 = \sum_{k=0}^{5} (-2x)^k = \frac{1-(-2x)^6}{1-(-2x)} = \frac{1-64x^6}{1+2x}$$

2. (a) $0, 0, 0, ...$
   (b) $0, 1, 0, 0, 0...$
   (c) $4, 3, -10, 55.$
   (d) $-64, 144, -108, 27.$
   (e) $0, 3, 3, 3, 3, ...$
   (f) $1, 6, 27, 108, ...$

3. (a) 1.

(b) 24.

(c) 1.

(d) 57 915.

4. 3 246.

5. (a) 120.

   (b) 48.

   (c) 21.

6. 14.

7. $\frac{1}{8} + \frac{1}{4}\binom{n+1}{1} + \frac{1}{2}\binom{n+2}{2} + \frac{1}{8}(-1)^n$.

8. 336.

9. 495.

10. (a)

$$g(x) = (1 + x^2 + x^4 + ...)(x^3 + x^5 + x^7 + ...)(x^4 + x^6).$$

   (b) i. 9.

   ii. 0.

11. (a)

$$g(x) = \frac{x^6(1 - x^4)}{(1 - x)^4}.$$

(b)
$$g(x) = \frac{(1-x^6)(1-x^3)}{(1-x)^2(1-x^2)^2}.$$

(c)
$$g(x) = x^{10}\frac{1}{(1-x)^4}.$$

(d)
$$g(x) = \frac{1}{(1-x^2)(1-x^3)^2(1-x)}.$$

12. The coefficient of x^{50} in,
$$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}}.$$

13. (a) 8.
    (b) 90.

14. (a)
$$g(x) = (1 + x + x^2 + x^3 + ...)^3.$$
    (b)
$$g(x) = x^3(1 + x + x^2 + x^3 + ...)^3.$$

15. 6.

5.3 Partitions of Integers

Solutions:

1. A partition on a positive integer n is a collection of unordered positive integers that all sum to n.

2. Generating functions are helpful in determining the number of possible partitions of integers as we can use generating functions to represent the number of summands of each possible size. This makes finding the total number of possible summands much easier.

3. A Ferrers diagram is a visual representation of a partition of some integer n where each summand is represented by a vertical row of dots, and the rows are organized from largest to smallest moving from left to right. In a Ferrers diagram for an integer n, there will be n total dots.

4. One example:
$$54 = 1 + 10 + 20 + 5 + 8 + 7 + 3$$

5.
$$\begin{aligned}
5 &= 1 + 1 + 1 + 1 + 1 \\
5 &= 1 + 1 + 1 + 2 \\
5 &= 1 + 2 + 2 \\
5 &= 1 + 1 + 3 \\
5 &= 1 + 4 \\
5 &= 2 + 3 \\
5 &= 5
\end{aligned}$$

6.

$$g(x) = (1 + x + x^2 + ...)^5$$

.

7. (a)

$$g(x) = x^k \prod_{i=1}^{k} \frac{1}{(1 - x^i)}.$$

(b)

$$g(x) = x^{2k+1} \prod_{i=1}^{k+1} \frac{1}{1 - x^{2i-1}}.$$

(c)

$$g(x) = \prod_{i=0}^{\infty} (1 + x)^{(2i+1)}.$$

(d)

$$g(x) = x^2 \prod_{i=1}^{\infty} \frac{1}{1 - x^i}.$$

(e)

$$g(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i},$$

where $i \not\equiv 0 \pmod 4$.

(f)

$$g(x) = \prod_{i=1}^{\infty} (1 + x^i).$$

(g)

$$g(x) = \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + x^{3i} + x^{4i} + x^{5i}).$$

(h)

$$g(x) = \prod_{i=1}^{12} (1 + x^i + x^{2i} + x^{3i} + x^{4i} + x^{5i}).$$

8. *Hint*: Find the two generating functions and explain why they're equal.

9. *Hint*: Compare the relevant generating functions.

10. *Hint*: Find a one-to-one correspondence between any partitioning of n and any partitioning of $2n$ into n parts.

11. *Hint*: Find a one-to-one correspondence between the two sets.

6 Recurrence Relations

6.1 First-Order Linear Recurrence Relations

Solutions:

1. A recurrence relation is an expression for a function $f(n)$ that is defined in terms of previous terms, such as $f(n-1)$, with one or more initial values for $f(k)$ stated.

2. Solving a recurrence relation means determining a function, whose domain is the set of non-negative integers, that describes the recurrence relation for all $n \geq 0$ without solving for previous terms.

3. $a_6 = -262$.

4. $a_n = 5 \cdot (-2)^n$, for $n \geq 0$.

5. $a_n = 909(\frac{1}{3})^n$, for $n \geq 0$.

6. $a_n = k \cdot (\frac{-6}{5})^n$.

7. $a_n = \frac{1296}{2401}(\frac{7}{2})^n$.

8. \$1668.25.

9. $b_n = 25 \cdot 3^n$, for $n \geq 0$.

Hint: Make the substitution $b_n = a_n^2$.

10. (a) $a_0 = 0$, and $a_{n+1} = a_n + 2n$, for $n \geq 1$.

    (b) $a_0 = 7$, and $a_{n+1} = \frac{2 \cdot a_n}{5}$, for $n \geq 1$.

11. $d = \frac{2}{7}$.

12. $5 \cdot (3^{36})$.

13. (a) $a_n = (-5)^n$, $n \geq 0$.

    (b) $a_n = (4)^{n-1}$, $n \geq 1$.

6.2 Second Order Linear Homogeneous Recurrence Relations with Constant Coefficients

Solutions:

1. Two initial values.

2. $C_0r^2 + C_1r - C_2 = 0.$

3. (a) $a_n = 19(4^n) - 14(5^n),\ \text{for } n \geq 0.$

   (b) $a_n = -2sin(\frac{\pi \cdot n}{2}),\ \text{for } n \geq 0.$

   (c) $a_n = 7(-3)^n + 41n(-3)^{n-1}.$

   (d) $a_n = (1+i)^n + (1-i)^n \text{ for } n \geq 0.$

   (e) $a_n = -\sqrt{3}^n + \frac{n(3+5\sqrt{3})}{3}\sqrt{3}^n,\ \text{for } n \geq 0.$

   (f) $a_n = -2(3^n) + 4^{n+1} \text{ for } n \geq 0.$

   (g) $a_n = \frac{-3i}{2}(3+i)^n + \frac{3i}{2}(3-i)^n \text{ for } n \geq 0.$

   (h) $a_n = -3(2^n) + 7n(2^n) \text{ for } n \geq 0.$

   (i) $a_n = \frac{5}{2}(2^n) + \frac{1}{2}(-2)^n \text{ for } n \geq 0.$

4. $b = \frac{-148}{53}$, $c = 153$.

$$a_n = (-1 - \frac{412}{2\sqrt{5423}})(\frac{74 - \sqrt{5423}}{53})^n + (\frac{413}{2\sqrt{5423}} - 1)(\frac{74 + \sqrt{5423}}{53})^n \text{ for } n \geq 0.$$

5. $3a_n = 5a_{n-1} - 11a_{n-2}$ for $n \geq 2$. The initial conditions are $a_0 = a, a_1 = b$ for any $a, b \in \mathbb{R}$.

6. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, with $a_0 = 0, a_1 = 1$.

   $a_n = \frac{\sqrt{5}}{5}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$ for $n \geq 0$.

7. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ with $a_1 = 2$ and $a_2 = 3$.

8. $c_1 = 9$, $c_2 = -18$.

9. $a_n = 3a_{n-1} - 2a_{n-2}$.

   $a_n = c_1 + c_2 2^n$, for $n \geq 0$.

10. $o_n = 3^n - 2^n$.

11. $a_n = \frac{7}{4}3^n - \frac{3}{4} - \frac{n}{2}$.

12. (a) $u_n = 2u_{n-1} + 2u_{n-2}$, where $u_0 = 1$.

(b) $u_n = 2u_{n-2} + u_{n-1}$, where $u_0 = 1$.